Economic Growth Continued: From Solow to Ramsey…

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Choosing a National Savings Rate

What can we say about economic policy and long-run growth? To keep matters simple, let us assume that the government can—by proper fiscal and monetary policies—set and keep the economy's savings-investment rate s at whatever level it wishes. What level should the government choose for the economy's savings rate?

It seems reasonable to assume that the government's objective is to maximize the well-being of the individuals who make up the society by maximizing the amount of goods and services that they can consume. Let us, for the moment, simplify things further and say that consumption C is equal to total production Y minus investment I:

$$C_t = Y_t - I_t$$

Where investment I is equal to the savings rate s times total production Y:

$$I_t = s \times Y_t$$

So consumption per worker C/L is equal to:

$$\frac{C_t}{L_t} = (1-s)Y_t$$

If we focus our attention on steady-states only, steady-state consumption per worker on the long-run growth path is equal to:

$$\frac{C_t}{L_t} = (1-s)\left(\frac{s}{n+g+\delta}\right)^{\left(\frac{\alpha}{1-\alpha}\right)} \times E_t$$

Maximizing Steady-State Consumption per Worker

What level of the savings rate s should the government choose if it wishes the economy to be on the long-run growth path that has the highest level of consumption per worker?

If we look at the rate of change—the derivative—of consumption per worker as a function of the savings rate:

$$\frac{d}{ds}\left(\frac{C_t}{L_t}\right) = \left(-(s)^{\left(\frac{\alpha}{1-\alpha}\right)} + \left(\frac{\alpha}{1-\alpha}\right)(1-s)(s)^{\left(\frac{2\alpha-1}{1-\alpha}\right)}\right) \times \left(\frac{E_t}{(n+g+\delta)^{\left(\frac{\alpha}{1-\alpha}\right)}}\right)$$

$$\frac{d}{ds}\left(\frac{C_t}{L_t}\right) = \left(-1 + \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-s}{s}\right)\right) \times (s)^{\left(\frac{\alpha}{1-\alpha}\right)}\left(\frac{E_t}{(n+g+\delta)^{\left(\frac{\alpha}{1-\alpha}\right)}}\right)$$

The rate of change of consumption per worker is zero—and the level of consumption per worker is at its highest—when:

$$s = \alpha$$

This savings rate is called the "golden rule" savings rate. The associated steady-state growth path is called the "golden rule" steady-state growth path.

Another way to look at the golden rule steady-state is to look at the marginal product of capital—the amount by which an additional unit of capital boosts output. The marginal product of capital is:

$$\frac{dY}{dK} = \alpha\frac{Y}{K}$$

Which is, at the steady-state growth path with K/Y = s/(n+g+δ) = α/(n+g+δ), equal to:

$$\frac{dY}{dK} = \alpha\frac{(n+g+\delta)}{\alpha} = n+g+\delta$$

At the golden rule steady-state growth path, the marginal product of capital is equal to the sum of the population growth rate, the efficiency of labor growth rate, and the depreciation rate.

To make the point another way, suppose that the economy starts with some steady-state capital-output ratio κ' and the government considers taking steps to increase the savings rate to boost the capital stock by one unit. The amount by which the change increases production is the marginal product of capital—n+g+δ. But this increase in the capital stock increases the amount of savings needed to maintain the new, higher capital-output ratio: n is needed to keep up with population growth, g to maintain pace with the increased efficiency of labor, and δ to offset depreciation on the higher capital stock.

Thus when the marginal product of capital is equal to n+g+δ—and the savings-investment rate s is equal to α—then the extra output produced by an increase in the capital-output ratio is just equal to the increase in savings and investment required to maintain that extra increase in the capital-output ratio. When the capital-output ratio is less than the golden rule steady-state, an increase in the capital-output ratio raises output by more than the required increase in savings and investment: thus consumption per worker can increase. When the capital-output ratio is more than the golden rule steady-state, an increase in the capital-output ratio does not raise output by enough to offset the required increase in savings and investment: thus consumption per worker must fall.

Implications for Economic Policy

If an economy begins at a steady state with a higher capital-output ratio than the golden rule steady state, then consumption per worker can be increased by reducing the savings rate. A decline in the savings rate will boost the steady-state level of consumption per worker, and thus boost consumption per worker in the long run. Moreover, by cutting back on savings and increasing the funds available for consumption, consumption per worker can be increased in the short run as well.

If the economy begins at a steady state with a lower capital-output ratio than in the golden rule, then the government must take steps to raise the savings rate in order to reach the golden rule steady state. In the long run, this increase in the savings rate will boost the steady-state level of consumption per worker, and thus boost consumption per worker in the long run. However, the increase in the savings rate reduces the funds available for consumption in the short run. When the economy begins above the golden rule, reaching the golden rule produces higher consumption at all moments in time. But when the economy begins below the golden rule, reaching the golden rule requires reducing the level of consumption now and in the near future in order to boost consumption in the long run.

A government trying to consider whether to try to move the economy toward the golden rule steady state has to consider whether the long run boost to consumption outweighs the short run cut in consumption. The government must decide whether this tradeoff between the near future and the distant future is worthwhile. How can we figure this out? We need an explicit framework: a utilitarian framework—the Ramsey Model.

The Ramsey Model

Begin with an objective function—social-welfare or representative-agent utility, as a function of per-capita consumption over time:

$$\max \int_{t=0}^{\infty} e^{-\rho t}\left(\frac{c_t^{1-\theta}-1}{1-\theta}\right)$$

Two parameters: θ and ρ.

Consider increasing savings and then decreasing it in order to postpone a small amount of consumption for a small period of time, and then returning to the previously-scheduled path for consumption. Then we have, at the optimal plan:

$$-(\delta c)\left(c_t^{-\theta}\right) + (\delta c)\left(1 + r(\delta t)\right)\left(e^{-\rho(\delta t)}\right)\left(e^{-n(\delta t)}\right)\left(c_{t+\delta t}^{-\theta}\right) = 0$$

Solve the algebra:

$$\left(1 + r(\delta t)\right)\left(e^{-\rho(\delta t)}\right)\left(e^{-n(\delta t)}\right)\left(\frac{c_{t+\delta t}^{-\theta}}{c_t^{-\theta}}\right) = 1$$

$$\left(1 + r(\delta t)\right)\left(e^{-\rho(\delta t)}\right)\left(e^{-n(\delta t)}\right)\left(\frac{c_t + (\delta t)\frac{dc_t}{dt}}{c_t}\right)^{-\theta} = 1$$

$$\left(1 + r(\delta t)\right)\left(1 - (\rho + n)(\delta t)\right)\left(1 + \frac{1}{c_t}\frac{dc_t}{dt}(\delta t)\right)^{-\theta} = 1$$

$$\left(1 + r(\delta t)\right)\left(1 - (\rho + n)(\delta t)\right)\left(1 - \theta\frac{1}{c_t}\frac{dc_t}{dt}(\delta t)\right) = 1$$

And arrive at:

$$r - (\rho + n) = \frac{\theta}{c_t}\frac{dc_t}{dt}$$

This tells us that the government's fiscal policy should be to adjust the national savings rate so that per-capita consumption grows according to:

$$\frac{dc_t}{dt} = \frac{\left(r - (\rho + n)\right)c_t}{\theta}$$

What is r? r is the **social** marginal product of capital:

$$r_t = \frac{\partial Y_t}{\partial K_t} - \delta = \frac{\alpha Y_t}{K_t} - \delta$$

So the right thing for the government to do is to change spending and taxes until consumption is growing at a rate $(r-(\rho+n))/\theta$.

- What is r? 6% + 3% taxes + 3% labor rents + 3% externalities = 15%…
- What is ρ? Impatience? Or should ρ be $-n$? And what if we recognize that n is endogenous? There are unsolved problems in the theory of applied utilitarianism…

- What is θ? θ is somewhere between one and 3…

Solving the Ramsey Model

What do these equations imply for the time paths of the variables in our economic growth model? To solve the model, start with the first-order condition for utility maximization as a function of per-capita consumption in the Ramsey model:

$$\frac{1}{c_t}\frac{dc_t}{dt} = \frac{r-(\rho+n)}{\theta}$$

scale up to aggregate consumption:

$$\frac{1}{C_t}\frac{dC_t}{dt} = \frac{r-(\rho+n)}{\theta} + n$$

Substitute in for the marginal product of capital using the Cobb-Douglass production function:

$$\frac{1}{C_t}\frac{dC_t}{dt} = \frac{1}{\theta}\left[\alpha\frac{Y_t}{K_t} - \delta - (\rho+n)\right] + n$$

Now consider the evolution of the capital stock:

$$\frac{1}{K_t}\frac{dK_t}{dt} = \frac{Y_t}{K_t} - \frac{C_t}{K_t} - \delta$$

And combine it with the utility-maximization condition to get the evolution of the consumption-to-capital ratio:

$$\frac{1}{C_t/K_t}\frac{d(C_t/K_t)}{dt} = \frac{1}{C_t}\frac{dC_t}{dt} - \frac{1}{K_t}\frac{dK_t}{dt} = \frac{1}{\theta}\left[\alpha\frac{Y_t}{K_t} - \delta - (\rho+n)\right] + n - \left[\frac{Y_t}{K_t} - \frac{C_t}{K_t} - \delta\right]$$

$$\frac{1}{C_t/K_t}\frac{d(C_t/K_t)}{dt} = \left(\frac{\alpha}{\theta} - 1\right)\frac{Y_t}{K_t} + \left(1 - \frac{1}{\theta}\right)\delta - \frac{(\rho+n)}{\theta} + n + \frac{C_t}{K_t}$$

At this point we can either bull ahead through the model, or we can attempt to solve the model for a convenient-but interesting special case by noticing that if α = θ then Y/K vanishes from the equation above. Let's do the special case first.

Following William Smith (2006)

William Smith (2006) is the first to note that the Ramsey model, with the transversality condition imposed, collapses into a one-dimensional linear system if we impose $\alpha = \theta$ on the parameters. The first-order condition for the consumption-to-capital ratio is:

$$\frac{1}{C_t/K_t}\frac{d\left(C_t/K_t\right)}{dt} = -\frac{1-\theta}{\theta}\delta - \frac{(\rho+n)}{\theta} + n + \frac{C_t}{K_t}$$

because the Y/K terms cancel. Thus the only possible non-explosive and hence admissible solution has the consumption-to-capital ratio constant at:

$$\frac{C_t}{K_t} = \frac{1-\theta}{\theta}\delta + \frac{(\rho+n)}{\theta} - n$$

Recall our equation for the Solow-model evolution of the capital-output ratio, which tells us that the rate of change of the capital-output ratio is linear in the capital-output ratio itself and the savings rate:

$$\frac{d(K_t/Y_t)}{dt} = \frac{d\kappa_t}{dt} = (1-\alpha)(s_t - (n+g+\delta)\kappa_t)$$

If the consumption-to-capital ratio is constant, then the savings rate is linear in the capital-output ratio:

$$s_t = 1 - \frac{C_t}{Y_t} = 1 - \frac{C_t}{K_t}\kappa_t = 1 - \left(\frac{1-\theta}{\theta}\delta + \frac{(\rho+n)}{\theta} - n\right)\kappa_t$$

And so we are down to a one-dimensional linear system, which we can solve exactly:

$$\frac{d(K_t/Y_t)}{dt} = \frac{d\kappa_t}{dt} = (1-\alpha)\left[1 - \left(\frac{1-\theta}{\theta}\delta + \frac{(\rho+n)}{\theta} - n\right)\kappa_t - (n+g+\delta)\kappa_t\right]$$

$$\frac{d\kappa_t}{dt} = (1-\alpha)\left[1 - \left(\frac{\delta+\rho+n+\theta g}{\theta}\right)\kappa_t\right]$$

$$\kappa_t = \kappa_0 + \left[\frac{\theta}{\delta+n+\rho+\theta g} - \kappa_0\right]\exp\left(-\left(\frac{(1-\alpha)(\delta+n+\rho+\theta g)}{\theta}\right)t\right)$$

And the complete solution to the model is:

$$L_t = L_0 \exp(nt)$$
$$E_t = E_0 \exp(gt)$$
$$Y_t / L_t = E_t (\kappa_t)^{\frac{\alpha}{1-\alpha}}$$
$$K_t / L_t = E_t (\kappa_t)^{\frac{1}{1-\alpha}}$$
$$C_t / L_t = (C_t / K_t)(K_t / L_t) = \left(\frac{1-\theta}{\theta}\delta + \frac{\rho + n}{\theta} - n \right) E_t (\kappa_t)^{\frac{1}{1-\alpha}}$$
$$s_t = 1 - C_t / Y_t = 1 - \left(\frac{1-\theta}{\theta}\delta + \frac{\rho + n}{\theta} - n \right) \kappa_t$$

with balanced growth-path values:

$$\kappa^* = \frac{\theta}{\delta + n + \rho + \theta g}$$
$$(C_t / K_t)^* = \left(\frac{1-\theta}{\theta}\delta + \frac{\rho + n}{\theta} - n \right)$$
$$s^* = 1 - (C_t / Y_t)^* = 1 - \left(\frac{(1-\theta)\delta + \rho + n - \theta n}{\theta} \right) \left(\frac{\theta}{\rho + n + \delta + \theta g} \right)$$
$$s^* = 1 - \left(\frac{(1-\theta)\delta + n + \rho - \theta n}{\rho + n + \delta + \theta g} \right) = \frac{\theta(n + g + \delta)}{\rho + n + \delta + \theta g}$$

Compare this to the Golden Rule values:

$$\kappa_{gr}^* = \frac{\alpha}{n + g + \delta}$$
$$s_{gr}^* = \frac{\alpha}{n + g + \delta}$$

Let's consider some sample parameters: $\alpha = \theta = 2/3$, $\delta = 0.05$, $n = 0.01$, $g = 0.03$, $\rho = 0.03$:

$$\kappa^* = \frac{\theta}{\delta + n + \rho + \theta g} = 6.06$$
$$(C_t / K_t)^* = \left(\frac{1-\theta}{\theta}\delta + \frac{\rho}{\theta} + \frac{1-\theta}{\theta} n \right) = .075$$
$$s^* = 1 - (C_t / Y_t)^* = 1 - \left(\frac{(1-\theta)\delta + n + \rho - \theta n}{\theta} \right) \left(\frac{\theta}{\rho + n + \delta + \theta g} \right)$$
$$s^* = 1 - \left(\frac{(1-\theta)\delta + \rho - \theta n}{\rho + \delta + \theta g} \right) = \frac{\theta(n + g + \delta)}{\rho + n + \delta + \theta g} = \frac{.06}{.11} = .545$$

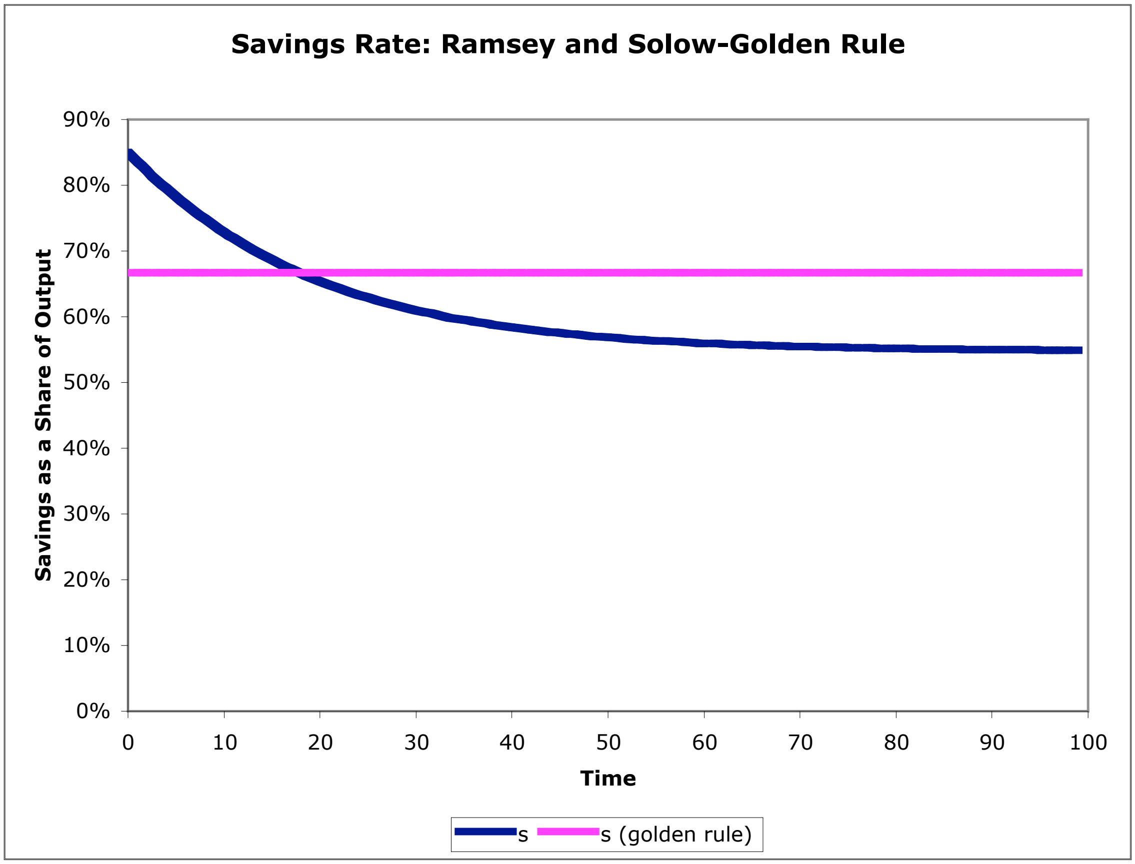
Compare this to the Golden Rule values:

$$\omega^* = .045$$

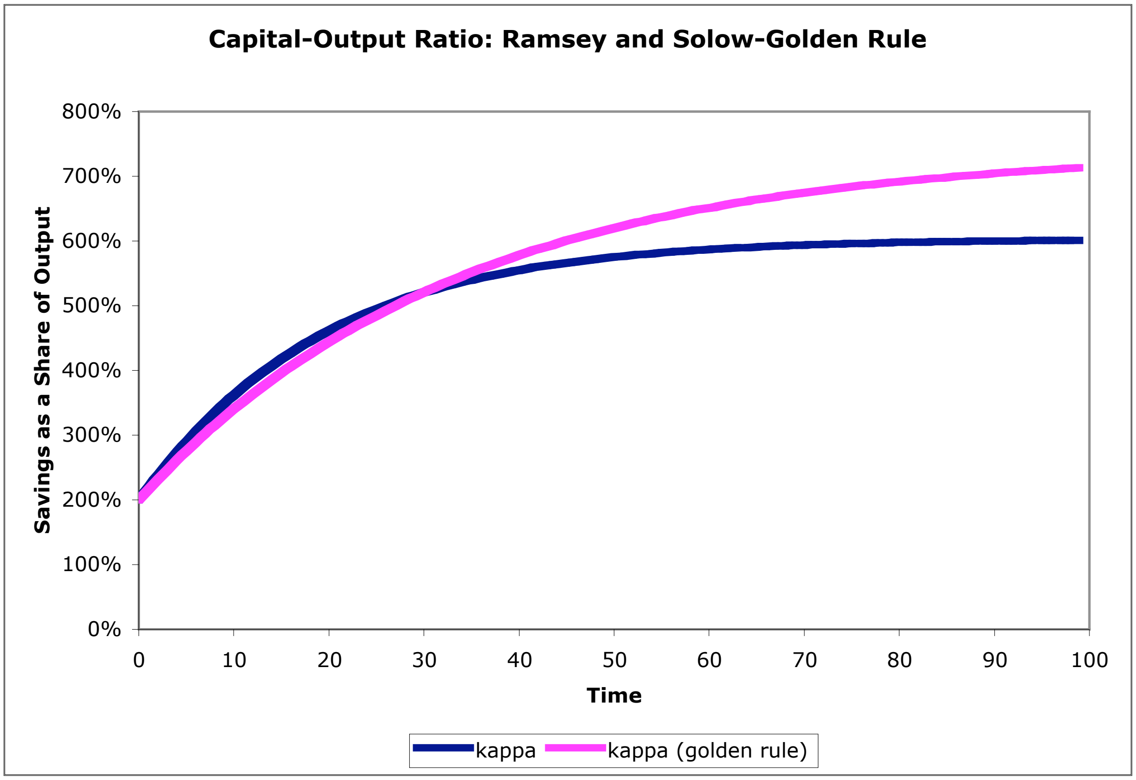
$$\kappa_{gr}^* = \frac{\alpha}{n + g + \delta} = 7.41$$

$$s_{gr}^* = \alpha = .67$$

Of particular interest is the behavior of the savings rate relative to its Golden Rule value:



And the behavior of the capital-output ratio:



Bulling Ahead

Alternatively, we can bull ahead. Let's analyze this model in terms of the two state variable pair made up of the capital-output ratio κ and the consumption-capital ratio, which we will call ω:

The laws of motion of these two variables are:

$$\frac{1}{\omega_t}\frac{d\omega_t}{dt} = \left[\omega_t - \left(\frac{1-\theta}{\theta}\delta + \frac{\rho}{\theta} + \frac{1-\theta}{\theta}n\right)\right] + \left(\frac{\alpha}{\theta} - 1\right)\frac{1}{\kappa_t}$$

$$\frac{d\kappa_t}{dt} = (1-\alpha)\left(1-(\omega_t + n + g + \delta)\kappa_t\right)$$

with steady-state values if we impose the transversality condition of:

$$\omega^*_t = \frac{(1-\alpha)\delta + (1-\alpha)n + (\theta-\alpha)g + \rho}{\alpha}$$

$$\kappa^*_t = \frac{\alpha}{[\delta + \theta g + \rho + n]}$$

The steady-state growth path for the model is:

$$L_t = L_0 \exp(nt)$$
$$E_t = E_0 \exp(gt)$$
$$Y^*_t / L_t = E_t \left(\frac{\alpha}{\delta + \theta g + \rho + n} \right)^{\frac{\alpha}{1-\alpha}}$$
$$K^*_t / L_t = E_t \left(\frac{\alpha}{\delta + \theta g + \rho + n} \right)^{\frac{1}{1-\alpha}}$$
$$C^*_t / L_t = \omega^* (K_t / L_t) = \left(\frac{1-\alpha}{\alpha}\delta + \frac{\rho}{\alpha} + \frac{\theta - \alpha}{\alpha} g + \frac{1-\alpha}{\alpha} n \right) \left(\frac{\alpha}{\delta + \theta g + \rho + n} \right)^{\frac{1}{1-\alpha}} E_t$$
$$s^*_t = \frac{\alpha\delta + \alpha g + \alpha n}{\delta + \theta g + \rho + n}$$

Rewriting the algebra around the steady-state produces the somewhat interesting form:

$$\frac{1}{\omega_t} \frac{d\omega_t}{dt} = (\omega_t - \omega^*_t) + \left(\frac{\alpha}{\theta} - 1 \right) \left(\frac{1}{\kappa_t} - \frac{1}{\kappa^*_t} \right)$$
$$\frac{d\kappa_t}{dt} = -\frac{(1-\alpha)(\delta + \theta g + \rho + n)}{\alpha} (\kappa_t - \kappa^*_t) - (1-\alpha)\left((\omega_t - \omega^*_t)(\kappa_t - \kappa^*_t) + \kappa^*_t (\omega_t - \omega^*_t) \right)$$

Compare this to the speed of convergence in the Solow model:

$$\frac{d\kappa_t}{dt} = -(1-\alpha)(n + g + \delta)(\kappa_t - \kappa^*_t)$$

The net rates-of-return in the Solow model along the Golden Rule balanced growth path and in the Ramsey model are:

$$r_{gr}^* = n + g$$
$$r_R^* = n + \rho + \theta g$$

Discussion

References:

William T. Smith (2006), "A Closed Form Solution to the Ramsey Model," *Contributions to Macroeconomics* 6:1 http://www.bepress.com/bejm/contributions/vol6/iss1/art3