# Ramsey-Cass-Koopmans Model I Handout

David Cass (1965), "Optimum Growth..." *Review of Economic Studies,* Tjalling Koopmans (1965), "On the Concept of Optimal Economic Growth," *The Economic Approach to Development Planning,* F.P. Ramsey (1928), "A Mathematical Theory of Saving," *Economic Journal*

**Basics:**

**Firms:**

**Have access to the production function Y=F(K, AL), which satisfies the same assumptions as in the Solow model (i.e., marginal product of capital gets arbitrarily large as the capital stock approaches zero, marginal product of capital gets arbitrarily small as the capital stock becomes arbitrarily large, factors of production have positive marginal products, production function exhibits diminishing returns, production function is constant returns to scale.**

**Firms hire labor and rent capital in competitive factor markets.**

**Households:**

**Own firms and capital. The household's utility function is:**

$$U_{t=0} = \int_{t=0}^{\infty} e^{-\rho t} u(C_t) \frac{L_t}{H} dt$$

**where H is the number of households. Note that as the "size" of the household grows over time, the flow of utility to this infinitely-lived household increases--people are utility machines, and the more people the better. Thus what we usually think of as "time preference" is not $\rho$, but $\rho$-n.**

**Most of the time the "instantaneous" utility function will be:**

$$u(C_t) = \frac{C^{1-\theta}}{1-\theta}, \theta > 0, \rho - n - g(1-\theta) > 0$$

**This is "constant relative risk aversion": the coefficient of relative risk aversion -Cu''(C)/u'(C) is equal to $\theta$, and is independent of C. Since there is no uncertainty in the model, why talk about risk aversion? Good question. Better to say that the elasticity of**

intertemporal substitution between consumption at any two points in time is 1/θ.

As θ goes to 0, the instantaneous utility function goes to ln(C). A useful first case to consider...

**Firm and household behavior.**

Firm behavior is uninteresting: at each point in time firms employ the (fixed at that moment) stocks of labor and capital, pay them their marginal products, and sell the resulting output. Firms earn zero profits.

Household behavior. Since the household has L/H members, its labor income flow is Aw*L/H, where w* is the real wage per unit of "effective labor"; [note difference of notation--I want to reserve w for the real real wage. The household's consumption is CL/H. Let's write:

$$R_t = \int_{s=0}^{t} r_s ds$$

for the total cumulative interest paid from time 0 to t: one unit of the good invested at time zero yields exp{R(t)} units at time t in the future.

Then the household's budget constraint is:

$$\int_{t=0}^{\infty} e^{-R_t} C_t \frac{L_t}{H} dt \leq \frac{K_0}{H} + \int_{t=0}^{\infty} e^{-R_t} A_t w*_t \frac{L_t}{H} dt$$

If we then divide everything by A(0)L(0)/H, substitute in the growth rates g and n of technology and population, and reduce everything to intensive form:

$$\int_{t=0}^{\infty} e^{-R_t} c_t e^{(n+g)t} dt \leq k_0 + \int_{t=0}^{\infty} e^{-R_t} w*_t \, e^{(n+g)t} dt$$

But do we really want all of these integrals? Isn't it better to rewrite the non-intensive form as:

$$\frac{K_0}{H} + \int e^{-R_t} [w*_t - c_t] A_t \frac{L_t}{H} dt \geq 0$$

or:

$$\lim_{s \to \infty} \left[ \frac{K_0}{H} + \int_{t=0}^{s} e^{-R_t} [w*_t - c_t] A_t \frac{L_t}{H} dt \right] \geq 0$$

To then say that the household's ownership of capital at any time is the current value of the difference between the cumulative totalsof what it has earned and what it has spent (so far):

$$\frac{K_s}{H} = e^{R_s}\frac{K_0}{H} + \int e^{[R_s - R_t]}[w*_t - c_t]\frac{A_t L_t}{H}\,dt$$

**And so write the budget constraint as:**

$$\lim_{s \to \infty}\left[e^{-R_s}\frac{K_s}{H}\right] \ge 0$$

$$\lim_{s \to \infty}\left[e^{-R_s}e^{(n+g)s}k_s\right] \ge 0$$

**The household cannot follow a consumption path that makes the present value of its asset holdings negative arbitrarily far into the future: no Ponzi schemes. Charles Ponzi.**

**Utility function in intensive form:**

**We know that:**

$$\frac{C^{1-\theta}}{1-\theta} = A_0^{1-\theta}e^{(1-\theta)gt}\frac{c_t^{1-\theta}}{1-\theta}$$

**We can then substitute this and the growth of the labor force into the utility function to get:**

$$U = \int_{t=0}^{\infty} e^{-\rho t}\left[A_0^{1-\theta}e^{(1-\theta)gt}e^{nt}\frac{c_t^{1-\theta}}{1-\theta}\right]\frac{L_0}{H}\,dt$$

$$U = B\int e^{-\beta t}\frac{c_t^{1-\theta}}{1-\theta}\,dt, B = A_0^{1-\theta}\frac{L_0}{H}, \beta = \rho - n - (1-\theta)g$$

**Now what do we do with this household utility function, in its intensive form?**

**Let's think of reducing the household's intensive consumption c at some moment t by a small amount $\delta$c, investing the fruits of this reduction in consumption for a small time $\Delta$t, and then increasing consumption at the end of this time interval in order to leave its capital holdings the same as in the original path of consumption taht we are thinking of.**

**If the household is optimizing, the marginal impact of this change on household utility must be negative, and of second order.**

**Our reduction in consumption of $\delta$c at time t allows an increase in consumption of:**

$$e^{[r_t - n - g]\Delta t}\delta c$$

**at time t+ $\Delta$t**

**The marginal utility of consumption at time t is:**

$$\frac{\partial U}{\partial c_t} = Be^{-\beta t}c_t^{-\theta}$$

**The marginal utility of consumption at time t+ $\Delta$t is:**

$$Be^{-\beta[t+\Delta t]}c_{t+\Delta t}^{1-\theta} = Be^{-\beta[t+\Delta t]}\left(c_t e^{\frac{dc_t}{dt}\frac{1}{c_t}\Delta t}\right)^{-\theta}$$

**In order for the household's path of consumption to be an optimum, the extra utility you lose by cutting consumption at time t must be balanced by the extra utility you gain by increasing consumption at time t+Δt. So:**

$$Be^{-\beta t}c_t^{-\theta}\delta c = Be^{-\beta[t+\Delta t]}\left(c_t e^{\frac{dc_t}{dt}\frac{1}{c_t}\Delta t}\right)^{-\theta} e^{[r_t-n-g]\Delta t}\delta c$$

$$Be^{-\beta t}c_t^{-\theta}\delta c = Be^{-\beta[t+\Delta t]}\left(c_t^{-\theta}\left(e^{\frac{dc_t}{dt}\frac{1}{c_t}\Delta t}\right)^{-\theta}\right) e^{[r_t-n-g]\Delta t}\delta c$$

$$1 = e^{-\beta[\Delta t]}\left(\left(e^{\frac{dc_t}{dt}\frac{1}{c_t}\Delta t}\right)^{-\theta}\right) e^{[r_t-n-g]\Delta t}$$

$$0 = -\beta\Delta t - \theta\frac{dc_t/dt}{c_t}\Delta t + [r_t-n-g]\Delta t = 0$$

$$\theta\frac{dc_t/dt}{c_t}\Delta t = [r_t-n-g-\beta]\Delta t$$

$$\frac{dc_t/dt}{c_t} = \frac{r_t-n-g-\beta}{\theta} = \frac{r_t-\rho-\theta g}{\theta}$$

**The so-called Euler equation.**

**Given c(0), and a time path for r(t), the Euler equation tells you how c must evolve over time in order for the consumption path to be an optimum.**

**The choice of c(0) is then determined by the requirement that the budget constraint be exhausted. Because the marginal utility of consumption is positive, if you find at the end of time that your capital holdings have a non-zero time-zero value, you aren't spending enough, and c(0) needs to be raised. If you find that you have negative capital holdings at the end of time, you are spending too much, and c(0) needs to be lowered.**

**Role played by CRRA utility.**

**In the log utility case (theta=1) our expressions become:**

$$\frac{1}{c_t} = e^{-\beta\Delta t} \frac{1}{c_t e^{\frac{dc_t/dt}{c_t}\Delta t}} e^{[r_t - n - g]\Delta t}$$

$$-\ln(c_t) = -\beta\Delta t - \ln(c_t) - \frac{dc_t/dt}{c_t}\Delta t - [r_t - n - g]\Delta t$$

**And dividing by Δt:**

$$\frac{dc_t/dt}{c_t} = r_t - n - g - \beta = r_t - \rho - \theta g$$

$$\frac{dc_t/dt}{c_t} = r_t - n - g - \beta = r_t - \rho - g$$

**What is happening to consumption per worker? Consumption per worker is growing at [r(t)-ρ]/θ. If the rate of return is higher than the rate at which the household discounts consumption, consumption per worker is growing. If the rate of return is lower, consumption per worker is shrinking...**

**Which of these are we going to use?**