New Growth Theory

Two ways to go: (I) think about the determinants of the effectiveness of labor--endogenize research and development; (II) break the link between capital's share of income and the importance of produced factors of production.

Let's do (I) first. We'll do it in two steps--first by saying that A is a function of K; then by considering A separately.

$$Y_t = K_t^{\alpha}\left[A_t L_t\right]^{1-\alpha}$$

$$A_t = BK_t^{\phi}$$

$$Y_t = K_t^{\alpha}\left[BK_t^{\phi} L_t\right]^{1-\alpha} = B^{1-\alpha} K_t^{\alpha+\phi(1-\alpha)} L_t^{1-\alpha}$$

$$\frac{dK_t}{dt} = sB^{1-\alpha} K_t^{\alpha+\phi(1-\alpha)} L_t^{1-\alpha}$$

in a model without depreciation.

The key thing to look at is then the exponent on K...

If phi is less than one, the long-run growth rate of this economy is a function of the rate of population growth, n. If phi is greater than one, there is explosive growth. If phi equals one, there is explosive growth if n is positive and steady growth if n is zero.

A second model:

Two sectors: a goods-producing sector and an R&D sector. A fraction a_L of the labor force in the R&D sector, a fraction a_K of the capital stock used in the R&D sector.

$$Y_t = \left[(1-a_K)K_t\right]^{\alpha}\left[A_t(1-a_L)L_t\right]^{1-\alpha}$$

$$\frac{dA_t}{dt} = B\left[a_K K_t\right]^{\beta}\left[a_L L_t\right]^{\gamma}\left[A_t\right]^{\theta}$$

Note that the production function for knowledge is not constant returns to scale in *anything*. Standard "replication" argument doesn't apply here.

Let's set:

$$\frac{dK_t}{dt} = sY_t$$

$$\frac{dL_t}{dt} = nL_t$$

To get some intuition, let's consider the model without capital:

Then the production functions for good and knowledge output become:

$$Y_t = A_t[(1-a_L)L_t]$$

$$\frac{dA_t}{dt} = B\left[a_L L_t\right]^{\gamma} A_t^{\theta}$$

Eliminating capital means that goods output per worker is proportional to A. So we can simply analyze A. The proportional growth rate of A is:

$$g_A = B a_L^{\gamma} L_t^{\gamma} A_t^{\theta-1}$$

$$\frac{dg_A}{dt} = \left[\gamma n + (\theta - 1) g_A\right] g_A$$

$$g_A{}^* = \frac{\gamma n}{1-\theta}$$

is the value of the technology growth rate at which technology is growing at a constant rate--if there is such a value.

Three cases: theta less than one, theta greater than one, theta equal to one.

Theta less than one case: The growth rate of g heads for a steady state value g^*_A; both Y/L and A then grow steadily at that rate. A balanced growth path.

Theta greater than one case. The growth rate of technology must be positive--and exploding as a function of technology growth. In fact, output reaches infinity in a finite amount of time; with theta greater than one and n=0, then the economy follows:

$$A_t = \frac{c_1}{(c_2 - t)^{1/(\theta-1)}}$$

$$c_1 = \frac{1}{\left[(\theta - 1) B a_L^{\gamma} L^{\gamma}\right]^{1/(\theta-1)}}$$

and there is a singularity at time c2.

Theta equal to one. Then everything simplifies to:

$$g_A = B a_L^{\gamma} L^{\gamma}$$

$$\frac{dg_A}{dt} = \gamma n g_A$$

Reintroducing capital:

$$Y_t = \left[(1-a_K)K_t\right]^{\alpha}\left[A_t(1-a_L)L_t\right]^{1-\alpha}$$

$$\frac{dA_t}{dt} = B\left[a_K K_t\right]^{\beta}\left[a_L L_t\right]^{\gamma}\left[A_t\right]^{\theta}$$

$$\frac{dK_t}{dt} = sY_t$$

$$\frac{dL_t}{dt} = nL_t$$

Let’s plug the first of these equations into the third to get:

$$\frac{dK_t}{dt} = s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}K_t^{\alpha}L_t^{1-\alpha}A_t^{1-\alpha}$$

$$g_K = \frac{1}{K_t}\frac{dK_t}{dt} = s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}\left[\frac{A_t L_t}{K_t}\right]^{1-\alpha}$$

This g(K) is rising if g(A)+n-g(K) is positive, falling if g(A)+n-g(K) is negative, constant if it is zero.

Similarly, dividing the knowledge accumulation equation by A(t) gives us an expression for the growth rate of A that looks rather similar:

$$g_A = Ba_K^{\beta}a_L^{\gamma}[K_t^{\beta}L_t^{\gamma}]A_t^{\theta-1}$$

Thus the behavior of G(A) depends on βg(K)+γn+(θ-1)g(A). The set of points where g(A) is constant has an intercept of -γn/β and a slope of (1-θ)/β.

Suppose the slope of the g(A) constant line is greater than one--suppose beta plus theta is less than one. Then the economy has a stable steady state constant growth-rate path. We can derive this path by: requiring that:

$$g_A* + n - g_K* = 0$$

$$\beta g_K* + \gamma n + (\theta - 1)g_A* = 0$$

Substituting:

$$\beta g_A * + (\beta + \gamma)n + (\theta - 1)g_A * = 0$$

$$g_A * = \frac{\beta + \gamma}{1 - (\theta + \beta)} n$$

$$g_K * = \frac{\beta + \gamma}{1 - (\theta + \beta)} n + n$$

Suppose beta plus theta is greater than one. Sooner or later the economy enters the region in which both growth rates are growing...

Endogenizing Saving:

What happens if we try to do for new growth theory models what we did for the Solow model in jumping to the Ramsey model--what happens if we try to endogenize savings?

A model with no population growth, and an infinitely-lived representative household:

$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{c_t^{1-\sigma}}{1-\sigma} dt$$

Capital and labor are paid their *private* marginal products. Households take their initial wealth and the path of interest rates and wages as given. And the production function is:

$$Y_t = K_t^{\alpha} \left[A_t L_t \right]^{1-\alpha}$$
$$A_t = BK_t$$

The production function for a single firm--indexed by i--is:

$$Y_{it} = B^{1-\alpha} K_t^{1-\alpha} K_{it}^{\alpha} L_{it}^{1-\alpha}$$

The marginal product of capital for a single firm is:

$$\alpha B^{1-\alpha} K_t^{1-\alpha} K_{it}^{\alpha-1} L_{it}^{1-\alpha}$$

In equilibrium, the capital-labor ratio *must* be equalized across firms, and must be equal ot the economy-wide aggregate capital-labor ratio. So:

$$r_t = \alpha B^{1-\alpha} K_t^{1-\alpha} K_t^{\alpha-1} L_t^{1-\alpha}$$
$$r_t = \alpha b$$

$$r_t = \bar{r}$$

where b is defined by:

$b = B^{1-\alpha} L^{1-\alpha}$

In a similar fashion, we can calcualte the real wage as the private marginal product of labor:

$$w_t = (1-\alpha) B^{1-\alpha} K_t^{1-\alpha} \left(K_{it}^{\alpha} L_{it}^{-\alpha} \right)$$
$$w_t = (1-\alpha) B^{1-\alpha} K_t^{1-\alpha} K_t^{\alpha} L^{-\alpha}$$
$$w_t = (1-\alpha) B^{1-\alpha} K_t^{\alpha} L^{-\alpha}$$
$$w_t = (1-\alpha) b \frac{K_t}{L}$$

The Ramsey model Euler equation tells us that:

$$\frac{1}{c_t} \frac{dc_t}{dt} = \frac{r_t - \rho}{\sigma} = \frac{\bar{r} - \rho}{\sigma} = \bar{g}$$

Now let's look for a steady-state growth path: we know consumption is growing at a rate g = (r-ρ)/σ, so let's hope for a solution in which capital and output are growing at the same rate...

Suppose the capital stock grows at a rate g. Then the real wage and the representative household's time-zero wealth--its present value of future income--are given by:

$$w_t = (1-\alpha) B \frac{K_0 e^{\bar{g}t}}{L}$$
$$\frac{W_0}{L} = \frac{K_0}{L} + \frac{1}{(\bar{r} - \bar{g})} (1-\alpha) b \frac{K_0}{L}$$

Since consumption grows at rate g, the present value of future lifetime consumption at time zero is:

$$\frac{c_0}{\bar{r} - \bar{g}}$$

The PV of lifetime consumption will equal the PV of lifetime income. So:

$$\frac{c_0}{\bar{r}-\bar{g}} = \frac{K_0}{L} + \frac{1}{(\bar{r}-\bar{g})}(1-\alpha)b\frac{K_0}{L}\; B\frac{K_0 e^{\bar{g}t}}{L}$$

$$c_0 = \left[\bar{r}-\bar{g}\right]\left[\frac{K_0}{L} + \frac{1}{(\bar{r}-\bar{g})}(1-\alpha)b\frac{K_0}{L}\right]$$

$$c_0 = \left[(1-\alpha)b + (\bar{r}-\bar{g})\right]\frac{K_0}{L}$$

$$c_0 = \left[(1-\alpha)b + (\alpha b-\bar{g})\right]\frac{K_0}{L}$$

$$c_0 = (b-\bar{g})\frac{K_0}{L}$$

or more generally:

$$c_t = (b-\bar{g})\frac{K_t}{L}$$

Now we know what c(0) must be if K and Y are growing at a rate g.Are K and Y growing at rate g? Yes.

In aggregate:

$$C_t = Lc_t = (b-\bar{g})K_t$$

Recalling our production function:

$$Y_t = K_t^{\alpha}\left[A_t L_t\right]^{1-\alpha} = K_t^{\alpha} B^{1-\alpha} K_t^{1-\alpha} L^{1-\alpha} = bK_t$$

and substituting it into our capital accumulation equation:

$$\frac{dK_t}{dt} = Y_t - C_t$$

gives:

$$\frac{dK_t}{dt} = Y_t - C_t = bK_t - (b-\bar{g})K_t = \bar{g}K_t$$

Thus consumption, capital, and output all grow at a constant rate in this equilibrium.