Suggested Solutions to Problem Set 2

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(a) Time-Averaging

(i) Notice that $c_{s+2} = c_{s+1} + e_{s+2} = c_s + e_{s+1} + e_{s+2}$, so

$$c_{s+2} - c_s = e_{s+1} + e_{s+2}.$$

Hence, the change in measured consumption from one two-period interval to the next can be written as

$$\begin{aligned} \Delta\bar{c}_1 &\equiv \frac{c_{t+2}+c_{t+3}}{2} - \frac{c_t + c_{t+1}}{2} = \frac{1}{2}\left(c_{t+2}-c_t\right) + \frac{1}{2}\left(c_{t+3}-c_{t+1}\right) \\ &= \frac{1}{2}\left(e_{t+1}+e_{t+2}\right) + \frac{1}{2}\left(e_{t+2}+e_{t+3}\right) = \frac{1}{2}e_{t+1} + e_{t+2} + \frac{1}{2}e_{t+3}. \end{aligned}$$

(ii) Similarly, the next change in measured consumption equals

$$\Delta\bar{c}_2 \equiv \frac{c_{t+4}+c_{t+5}}{2} - \frac{c_{t+2}+c_{t+3}}{2} = \frac{1}{2}e_{t+3} + e_{t+4} + \frac{1}{2}e_{t+5}.$$

Using the fact that the e_s are independent,

$$\begin{aligned} \operatorname{cov}\left(\Delta\bar{c}_1, \Delta\bar{c}_2\right) &= \operatorname{cov}\left(\frac{1}{2}e_{t+1} + e_{t+2} + \frac{1}{2}e_{t+3}, \frac{1}{2}e_{t+3} + e_{t+4} + \frac{1}{2}e_{t+5}\right) \\ &= \operatorname{cov}\left(\frac{1}{2}e_{t+3}, \frac{1}{2}e_{t+3}\right) = \frac{1}{4}\sigma^2, \end{aligned}$$

where σ^2 is the variance of the white noise e_s. If measured consumption $\bar{c}_k \equiv \frac{1}{2}\left(c_{t+2k} + c_{t+2k+1}\right)$ for $k = 0, 1, 2, ...$ were a random walk, we could write $\bar{c}_k = \bar{c}_{k-1} + u_k$, where u_k is white noise, so that $\Delta\bar{c}_k \equiv \bar{c}_k - \bar{c}_{k-1} = u_k$ and $\Delta\bar{c}_{k+1} = u_{k+1}$ would be uncorrelated. But, the change in measured consumption $\Delta\bar{c}_2$ from one two-period interval to the next and the previous change in measured consumption $\Delta\bar{c}_1$ appear to be correlated. Therefore, time-averaged measured consumption $\bar{c}_k$ does not follow a random walk.

(iii) If measured consumption in a two-period interval equals consumption in the second period, that is $\tilde{c}_k \equiv c_{t+2k+1}$ for $k = 0, 1, 2, \dots$, then

$$\Delta\tilde{c}_1 \equiv \tilde{c}_1 - \tilde{c}_0 = c_{t+3} - c_{t+1} = e_{t+2} + e_{t+3}$$

and

$$\Delta\tilde{c}_2 \equiv \tilde{c}_2 - \tilde{c}_1 = c_{t+5} - c_{t+3} = e_{t+4} + e_{t+5}.$$

As a result, $\text{cov}(\Delta\tilde{c}_1, \Delta\tilde{c}_2) = 0$ and measured consumption $\tilde{c}_k$ follows a random walk.

(b) Saving and Taxation To ensure that tax revenues are positive, we assume that $Y_1 \geq C_1^0$, where C_1^0 denotes optimal first-period consumption in the case of interest-income taxes.

(i) In the presence of proportional interest-income taxes, the consumer's budget constraint equals

$$C_1 + \frac{C_2}{1+r} \leq Y_1 + \frac{Y_2}{1+r} - t\frac{r(Y_1 - C_1)}{1+r}. \tag{1}$$

The left-hand-side is the present value of consumption; the right-hand-side equals the present value of income minus taxes on interest income. Multiplying both sides by $(1+r)$ and rearranging gives

$$C_1 + \frac{C_2}{1+(1-t)r} \leq Y_1 + \frac{Y_2}{1+(1-t)r}.$$

(ii) In the presence of lump-sum taxes, the consumer's budget constraint equals

$$C_1 + \frac{C_2}{1+r} \leq Y_1 + \frac{Y_2}{1+r} - T_1 - \frac{T_2}{1+r}, \tag{2}$$

where the right-hand-side equals the present value of income minus lump-sum taxes.

(iii) To leave the present value of government revenues unchanged, the lump-sum taxes T_1 and T_2 must satisfy

$$T_1 + \frac{T_2}{1+r} = t\frac{r(Y_1 - C_1^0)}{1+r}. \tag{3}$$

(iv) Denote the optimal consumption bundles with interest-income taxes and with lump-sum taxes by (C_1^0, C_2^0) and (C_1', C_2'), respectively. Optimal consumption satisfies the budget constraint with equality, presuming a strictly increasing utility function. Then, using the revenue-neutrality condition (3) and the budget constraints (1) and (2), we get

$$C_1^0 + \frac{C_2^0}{1+r} = Y_1 + \frac{Y_2}{1+r} - t\frac{r(Y_1 - C_1^0)}{1+r} = Y_1 + \frac{Y_2}{1+r} - T_1 - \frac{T_2}{1+r} = C_1' + \frac{C_2'}{1+r}.$$

As a result, the present value of optimal consumption is the same in both cases, so the optimal consumption bundle (C_1^0, C_2^0) under interest-income taxes is just affordable under lump-sum taxes.

(v) We just showed that the initial optimum (C_1^0, C_2^0) lies on both budget constraints. So, the change in taxes involves no income effect (using the so-called Slutsky decomposition); there is merely a substitution effect due to the increase in the relative price of current consumption from $1 + (1-t)r$ to $1 + r$. This reduces current consumption C_1 and increases future consumption C_2. The effect of a change from interest-income taxes to lump-sum taxes is illustrated in Figure 1. The budget line in the presence of interest-income taxes is denoted by B^0B^0. For $C_1 < Y_1$ it has slope $-[1 + (1-t)r]$; for $C_1 > Y_1$ there are no positive savings and therefore no interest-income taxes so that the slope equals $-(1+r)$. The endowment (Y_1, Y_2) is denoted by E. The budget line $B'B'$ in the presence of revenue-neutral lump-sum taxes goes through the initial optimum C^0 and has slope $-(1+r)$. The new optimum is denoted by C' and leads to lower first-period consumption. Clearly, the consumer is better off with lump-sum taxes.

(c) Mankiw's Durable Goods Model Let the quadratic utility function equal $u(C) = C - \frac{a}{2}C^2$. To prevent confusion between consumption expenditures at time t and the expectation as of t, denote the former by X_t. So, durable consumption services are given by

$$C_t = (1-b)C_{t-1} + X_t. \tag{4}$$

(i) A marginal reduction in X_t such that the combined changes in X_t, X_{t+1} and X_{t+2} leave the present value of consumption purchases, $\sum_{s=0}^{T} X_{t+s}$, unchanged satisfies

$$dX_t + dX_{t+1} + dX_{t+2} = 0. \tag{5}$$

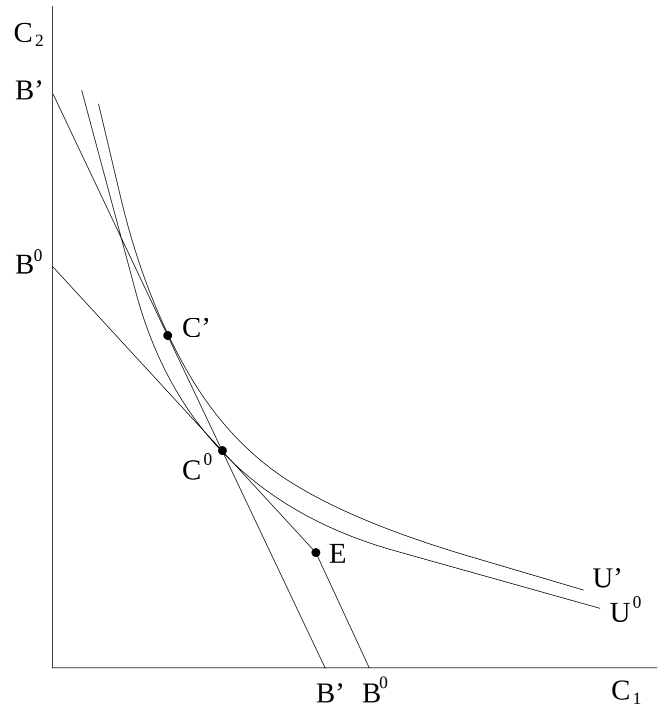

Figure 1: From interest-income taxes to lump-sum taxes

Using (4), the change in durable consumption services satisfies

$$dC_t = dX_t \quad (6)$$
$$dC_{t+1} = (1-b)dC_t + dX_{t+1} \quad (7)$$
$$dC_{t+2} = (1-b)dC_{t+1} + dX_{t+2} \quad (8)$$

The flow of durable consumption services will be constant from period $t+2$ on if $dC_{t+2} = 0$. As a consequences, substituting (6) and (7) into (8) yields

$$(1-b)^2 dX_t + (1-b)dX_{t+1} + dX_{t+2} = 0. \quad (9)$$

Thus, we have two equations, (5) and (9), to solve for $dX_{t+1}$and dX_{t+2}, given dX_t. Substituting $dX_{t+2} = -dX_t - dX_{t+1}$ from (5) into (9) and rearranging gives

$$dX_{t+1} = (b-2)\, dX_t, \quad (10)$$

and so,

$$dX_{t+2} = (1-b)\, dX_t.$$

(ii) Regarding the effect of these changes on the level of durable consumption services, we know that by construction, $dC_t = dX_t$ and $dC_{t+2} = 0$. Substituting (6) and (10) into (7) produces

$$dC_{t+1} = -dX_t. \tag{11}$$

The effect on utility at time $t+s$ equals $du(C_{t+s}) = u'(C_{t+s})dC_{t+s}$, so the effect on expected life-time utility, $U_t \equiv E_t\left[\sum_{s=t}^{T} u(C_{t+s})\right]$, is equal to

$$dU_t = E_t\left[\sum_{s=t}^{T} u'(C_{t+s})dC_{t+s}\right] = u'(C_t)dC_t + E_t\left[u'(C_{t+1})dC_{t+1}\right].$$

Substituting $u'(C) = 1 - aC$, (6) and (11),

$$dU_t = (1 - aC_t)\, dX_t - (1 - aE_t\left[C_{t+1}\right])\, dX_t. \tag{12}$$

(iii) It follows from (12) that marginal changes in spending dX_t have no first-order effect on expected utility ($dU_t = 0$) if

$$C_t = E_t[C_{t+1}]. \tag{13}$$

Similarly, we get

$$E_t\left[C_{t+1}\right] = E_t[C_{t+2}].$$

(iv) Define $e_{t+1} \equiv C_{t+1} - E_t[C_{t+1}]$, so that $E_t\left[e_{t+1}\right] = 0$, and use (13) to write

$$C_{t+1} = C_t + e_{t+1}. \tag{14}$$

Clearly, durable consumption services follow a random walk. The consumer smoothes the consumption of durables' services over time.

(v) Using (4), $X_t = C_t - (1-b)C_{t-1}$ and $X_{t+1} = C_{t+1} - (1-b)C_t$. So (14) implies

$$\begin{aligned} X_{t+1} - X_t &= C_{t+1} - C_t - (1-b)\left(C_t - C_{t-1}\right) \\ &= e_{t+1} - (1-b)e_t \equiv u_{t+1}. \end{aligned}$$

Hence, $X_{t+1} = X_t + u_{t+1}$, where $E_t\left[u_{t+1}\right] = -(1-b)e_t \neq 0$. Therefore, the level of consumption expenditures on durables does not follow a random walk. In particular, suppose the consumer decides to increase the level of durable consumption by one unit ($e_t = 1$) because of an unanticipated rise in life-time resources. Then this will lead to an increase in expenditures on durables by one unit ($u_t = 1$) this period. But next period, purchases of durable consumption goods drop ($u_{t+1} = -(1-b)$) so that the new level of expenditures just covers the extra amount of break-up (b) of the increase in consumption. Notice that in the special case in which $b = 1$, consumption goods are nondurable, so that consumption expenditures equal consumption services. In that case, both follow a random walk.

(d) Precautionary Saving The constant relative risk aversion (CRRA) utility function is $u(C) = C^{1-\theta}/(1-\theta)$, where $\theta > 0$ is the coefficient of relative risk aversion. Let Ω_t be the information available at time t, then the conditional log-normal distribution of C_{t+1} amounts to

$$\ln C_{t+1}|\Omega_t \sim N(E_t[\ln C_{t+1}], V),$$

where V is the conditional variance of $\ln C_{t+1}$. Applying the stochastic Euler equation (R 7.28), for a constant risk-free interest rate r and a subjective discount rate ρ,

$$C_t^{-\theta} = \frac{1+r}{1+\rho} E_t\left[C_{t+1}^{-\theta}\right]. \tag{15}$$

Notice that $C_{t+1}^{-\theta}$ also has a conditional log-normal distribution, since we have $\ln C_{t+1}^{-\theta}|\Omega_t \sim N(-\theta E_t[\ln C_{t+1}], \theta^2 V)$. Now we use the fact that for any normally distributed variable x with mean μ and variance σ^2, i.e. $x \sim N(\mu, \sigma^2)$, $E[e^x] = e^{\mu + \frac{1}{2}\sigma^2}$. For $x \equiv \ln X$ this implies that a variable X with a lognormal distribution such that $\ln X \sim N(\mu, \sigma^2)$, has expectation $E[X] = E\left[e^{\ln X}\right] = e^{\mu + \frac{1}{2}\sigma^2}$. Hence, we obtain

$$E_t\left[C_{t+1}^{-\theta}\right] = e^{-\theta E_t[\ln C_{t+1}] + \frac{1}{2}\theta^2 V}. \tag{16}$$

Writing (15) in logs and substituting (16),

$$-\theta \ln C_t = \ln(1+r) - \ln(1+\rho) - \theta E_t[\ln C_{t+1}] + \frac{1}{2}\theta^2 V.$$

Rearranging gives

$$\ln C_t = \frac{1}{\theta}[\ln(1+\rho) - \ln(1+r)] + E_t[\ln C_{t+1}] - \frac{1}{2}\theta V. \tag{17}$$

This establishes the relationship between consumption this period, C_t, and the log-normal distribution of C_{t+1} that is characterized by the conditional mean

$(E_t[\ln C_{t+1}])$ and variance (V) of $\ln C_{t+1}$. An increase in the variance V of next period's consumption reduces current consumption; the greater uncertainty about future consumption leads to higher precautionary savings. Notice that (17) implies that $\ln C_{t+1} = \alpha + \ln C_t + e_t$, where $\alpha \equiv \frac{1}{\theta}[\ln(1+r) - \ln(1+\rho)] + \frac{1}{2}\theta V$ and e_t is white noise. Hence, if future consumption is log-normal and utility is CRRA, then log consumption follows a random walk with drift.