

# Suggested Problem Set 3 Solutions

## Economics 202B – Second Half

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### 11.1 The stability of fiscal policy. (Blinder and Solow, 1973.)

Setup:

$$\begin{aligned}\delta(t) &\equiv \dot{D}(t), \text{ debt outstanding} \\ d(t) &= D(t)/Y(t), \text{ debt-to-output ratio} \\ \frac{\dot{Y}(t)}{Y(t)} &= g, \text{ growth rate of output}\end{aligned}$$

(a) The deficit-to-output ratio is constant:  $\delta(t)/Y(t) = a$ , where  $a > 0$ .

1. One can solve for  $\dot{d}(t)$  as follows:

$$\begin{aligned}\dot{d}(t) &= \left[ \dot{D}(t)Y(t) - D(t)\dot{Y}(t) \right] / Y(t)^2 \\ &= \left[ \delta(t)Y(t) - D(t)\dot{Y}(t) \right] / Y(t)^2 \\ &= a - gd(t).\end{aligned}\tag{1}$$

where the last line uses the definitions of the deficit-to-output ratio, the debt-to-output ratio, and output growth.

2. One can immediately see from (1) that the phase diagram will be straight downward-sloping line with a slope equal to  $g$  (see Figure 1.). The debt-to-output ratio is increasing when  $d < a/g$  and is decreasing when  $d > a/g$ . Therefore, the debt-to-output ratio always converges to equilibrium value  $a/g$ , so the system is stable.

(b) The ratio of the primary deficit-to-output is now a constant  $a$ , so the total deficit at  $t$  is now  $\delta(t) = aY(t) + r(t)D(t)$ .

1. One can solve for  $\dot{d}(t)$  as follows:

$$\begin{aligned}\dot{d}(t) &= \left[ \dot{D}(t)Y(t) - D(t)\dot{Y}(t) \right] / Y(t)^2 \\ &= \left[ \delta(t)Y(t) - D(t)\dot{Y}(t) \right] / Y(t)^2 \\ &= \left[ (aY(t) + r(d(t))D(t))Y(t) - D(t)\dot{Y}(t) \right] / Y(t)^2 \\ &= a + [r(d(t)) - g]d(t).\end{aligned}\tag{2}$$

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\*I thank Andrea De Michelis for allowing me to use and edit the solution to 11.4 below.

2. First consider when  $a$  is sufficiently small so that  $\dot{d}(t)$  is negative for some values of  $d(t)$ . Since  $r(d(t))$  is a convex function, so will be  $\dot{d}(t)$  and there will be two points where  $\dot{d}(t) = 0$  (see Figure 2.). Let these two points be  $d_1$  and  $d_2$ , where  $d_1 < d_2$ . One can immediately see that  $d_1$  is a stable equilibrium while  $d_2$  is unstable. Therefore, if the debt-to-output ratio starts at any point below  $d_2$  it will converge to  $d_1$ , remain stable if it begins at  $d_1$ , and otherwise will explode.

Second consider when  $a$  is sufficiently large so that  $\dot{d}(t)$  is always positive for all values of  $d(t)$ . Since  $r(d(t))$  is a convex function, so will be  $\dot{d}(t)$ , and in this case the debt-to-output ratio will always be rising, regardless of where it begins. See Figure 3.

Finally note that it might be possible that  $\dot{d}(t)$  is equal to zero for only one value of  $d(t)$  (call this point  $d_1$ ), so the system will only have one equilibrium. In this case (not shown), the debt-to-output ratio will converge to  $d_1$  if it begins at a point below or equal to it, and explode otherwise.

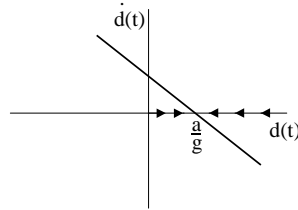


Figure 1.

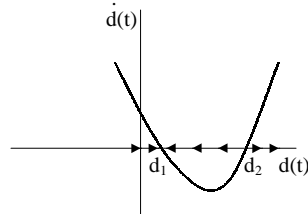


Figure 2.

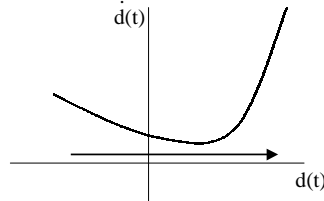


Figure 3.

## 11.4 Barro tax-smoothing model.

Using dynamic programming, one can derive an expression for the expected present value of the revenue the government must raise when  $G = G_H$ :

$$V_H(\Delta t) = \int_{t=0}^{\Delta t} [e^{-rt} e^{-at} (G_H + rD)] dt + e^{-r\Delta t} [e^{-a\Delta t} V_H(\Delta t) + (1 - e^{-a\Delta t}) V_L(\Delta t)].$$

By algebraic manipulations:

$$\begin{aligned} V_H(\Delta t) &= \frac{G_H + rD}{-a - r} (e^{-(a+r)\Delta t} - 1) + e^{-r\Delta t} [e^{-a\Delta t} V_H(\Delta t) + (1 - e^{-a\Delta t}) V_L(\Delta t)] \\ &= \frac{\frac{G_H + rD}{-a - r} (e^{-(a+r)\Delta t} - 1) + e^{-r\Delta t} (1 - e^{-a\Delta t}) V_L(\Delta t)}{1 - e^{-(a+r)\Delta t}} \\ &= \frac{G_H + rD}{a + r} + e^{-r\Delta t} \frac{1 - e^{-a\Delta t}}{1 - e^{-(a+r)\Delta t}} V_L(\Delta t). \end{aligned}$$

Taking the limit as  $\Delta t \rightarrow 0$ :

$$\begin{aligned} V_H &= \frac{G_H + rD}{a + r} + \frac{a}{a + r} V_L \\ &= \frac{G_H + rD + aV_L}{a + r}, \end{aligned}$$

which can be rewritten as

$$rV_H = (G_H + rD) + a(V_L - V_H). \quad (3)$$

Note that the above equation can be interpreted as an asset pricing condition for a risk-neutral investor.

Similarly, one can derive an expression for the expected present value of the revenue the government must raise when  $G = G_L$ :

$$V_L = \frac{1}{b + r} (G_L + rD + bV_H)$$

or

$$rV_L = (G_L + rD) + b(V_H - V_L). \quad (4)$$

Solving (3) and (4) for  $V_L$  and  $V_H$ :

$$V_L = \frac{aG_L + rG_L + bG_H}{r(a + b + r)} + D, \quad (5)$$

$$V_H = \frac{aG_L + rG_H + bG_H}{r(a + b + r)} + D. \quad (6)$$

The above expressions give the expected present value of the revenue the government must raise as a function of the its expenditures, the amount of debt outstanding and the parameters of the model. Since we are in a Barro (1979) world in which output is constant and the cost function is quadratic, the government tries to keep the level of tax constant over time. For example, if at time 0 the economy is in the high state so taxes are  $T_H$ , since taxes follows a random walk one can show that  $E_t(T) = T_H$  for  $t \geq 0$ . Thus, when  $G$  equals  $G_H$ , the government plans to impose a constant tax  $T_H$  forever such that

$$\int_0^{\infty} e^{-rt} T_H dt = V_H.$$

Solving the above integral and substituting eq. (6) for  $V_H$  yields

$$T_H = \frac{aG_L + rG_H + bG_H}{a + b + r} + rD. \quad (7)$$

The above equation gives an expression for taxes at a given time when the level of public expenditures is high. Note that  $T_H$  is not time invariant if  $\dot{D} \neq 0$ .

Similarly, when  $G$  equals  $G_L$ , the government sets  $T_L$  equal to

$$T_L = \frac{aG_L + rG_L + bG_H}{a + b + r} + rD. \quad (8)$$

As pointed out above, the path of taxes during an interval in which  $G$  is constant is driven by the path of the outstanding debt. In general, the budget deficit equals

$$\dot{D} = G - T + rD.$$

Thus, the path of taxes in an interval in which  $G$  equals  $G_H$  is given by

$$\begin{aligned} \dot{T}_H &= r\dot{D} \\ &= r(G_H - T_H + rD) \\ &= r \frac{a(G_H - G_L)}{a + b + r}. \end{aligned}$$

Since  $G_H > G_L$ , the above expression tells us that taxes are increasing over time as long as  $G$  equals  $G_H$ . The intuition is that the government know that its expenditures will go down in the future, and so it prefers to run a deficit in order to smooth taxes over time. However, as long as its expenditures do not drop to  $G_L$ , the expected present value of the revenue the government must raise to satisfy its budget constraint will keep on increasing since the switch has not yet occurred and the debt is raising.

At the moment when  $G$  falls to  $G_L$ , taxes will drop from  $T_H$  to  $T_L$ . Formally:

$$\begin{aligned}\dot{T}_{JUMP} &= T_L - T_H \\ &= -\frac{r(G_H - G_L)}{a + b + r}.\end{aligned}$$

The path of taxes in an interval in which  $G$  equals  $G_L$  is again driven by the path of the outstanding debt:

$$\begin{aligned}\dot{T}_L &= r\dot{D} \\ &= r(G_L - T_L + rD) \\ &= r\frac{b(G_L - G_H)}{a + b + r}.\end{aligned}$$

Since  $G_H > G_L$ , the above expression tells us that taxes are falling over time as long as  $G$  equals  $G_H$ . The intuition is that the government know that its expenditures will go up in the future, and so it prefers to run a surplus in order to smooth taxes over time. However, as long as its expenditures do not rise to  $G_H$ , the expected present value of the revenue the government must raise to satisfy its budget constraint will keep on decreasing since the switch has not yet occurred and the debt is falling.

### 11.7 Tabellini-Alesina model with initial debt.

Setup:

$$\begin{aligned}M_1 + N_1 &= D \\ M_2 + N_2 &= W - D - D_0, \text{ where } D_0 \text{ is initial debt} \\ \alpha^{MED} &= 0 \text{ or } 1 \text{ in either period}\end{aligned}$$

The solution to this problem follows the steps taken in Section 11.6 in solving the Tabellini-Alesina model. The only difference is that one must take into account the repayment of the initial debt,  $D_0$ , in the policymaker's second period budget constraint. One could assume that  $D_0$  is paid off by the first period policy-maker, but given the strategic nature of the game it is hard to justify this assumption. Therefore, equations (11.26) and (11.27) are replaced by

$$\begin{aligned}\frac{U'(W + D)}{U'(W - D - D_0)} &= \pi, \text{ if } \alpha^{MED} = 1 \text{ in period 1, and} \\ \frac{U'(W + D)}{U'(W - D - D_0)} &= 1 - \pi, \text{ if } \alpha^{MED} = 0 \text{ in period 2,}\end{aligned}$$

respectively. We see therefore, that the qualitative results discussed in the text follow vis-à-vis the incentives for the first period policymaker to accumulate debt. However, since there already exists some initial debt, the first period policymaker does not have to accumulate as much debt in order to constrain a potentially different period-2 policymaker's spending. To see this formally, differentiate either of the first-order conditions to arrive at

$$\frac{\partial D}{\partial D_0} = -\frac{U'(W + D)U''(W - D - D_0)}{U''(W + D)U'(W - D - D_0) + U'(W + D)U'''(W - D - D_0)} < 0.$$