Math-Heavy Appendix to Chapter 4:

How Fast Does the Economy Head Toward Its Steady-State Balanced-Growth Path?

(Advanced: Calculus Intensive)

The body of Chapter 4 asserts that in the Solow growth model an economy that is not on its steady-state balanced-growth path is heading towards it. This appendix proves that assertion, and in the process derives how fast the economy heads towards its steady-state balanced-growth path. This is a calculus-heavy appendix: it first uses calculus to derive an expression for the growth rate of output, second uses calculus to derive an expression for the growth rate of the capital stock, third uses calculus to derive an expression for the proportional rate of change of the capital-output-ratio, and fourth uses advanced calculus—differential equations—to solve and then analyze the equation for the proportional rate of change of the capital-output ratio.

If you are good at math (and have the right background), you will find that this appendix makes a substantial part of Chapter 4 clearer by filling in details and replacing hand-waving by clearer arguments. If not… then not: this appendix is likely to create confusion even where none existed before, and worsen already-existing confusion.

The Rate of Change of Output Y

If we rewrite the production function:

$$\frac{Y_t}{L_t} = \left(\frac{K_t}{L_t}\right)^{\alpha} \times E_t^{1-\alpha}$$

as:

$$Y_t = K_t^{\alpha} \times L_t^{1-\alpha} \times E_t^{1-\alpha}$$

by multiplying both sides of the equation by the labor force L, then in this form it is easy to calculate how much output changes in response to changes in the capital stock, in the labor force, and in the efficiency of labor.

We use the calculus notation dy/dx to stand for how much a variable y changes in response to a change in the variable x, and the basic calculus principle that if:

$$y = x^z \times A$$

then:

$$\frac{dy}{dx} = z \times x^{z-1} \times A$$

The answers to how much output changes in response to changes in its determinants are then straightforward:

$$\frac{dY_t}{dK_t} = \alpha K_t^{\alpha-1} \times L_t^{1-\alpha} \times E_t^{1-\alpha} = \alpha \frac{Y_t}{K_t}$$

$$\frac{dY_t}{dL_t} = K_t^{\alpha} \times (1-\alpha) L_t^{-\alpha} \times E_t^{1-\alpha} = (1-\alpha)\frac{Y_t}{L_t}$$

$$\frac{dY_t}{dK_t} = K_t^{\alpha} \times L_t^{1-\alpha} \times (1-\alpha) E_t^{-\alpha} = (1-\alpha)\frac{Y_t}{E_t}$$

Recall the *chain rule* calculus principle for determining how a variable changes over time depending on how rapidly its determinants are changing:

$$\frac{dY_t}{dt} = \left(\frac{dY_t}{dK_t}\right)\frac{dK_t}{dt} + \left(\frac{dY_t}{dL_t}\right)\frac{dL_t}{dt} + \left(\frac{dY_t}{dE_t}\right)\frac{dE_t}{dt}$$

$$\frac{dY_t}{dt} = \left(\alpha\frac{Y_t}{K_t}\right)\frac{dK_t}{dt} + \left((1-\alpha)\frac{Y_t}{L_t}\right)\frac{dL_t}{dt} + \left((1-\alpha)\frac{Y_t}{E_t}\right)\frac{dE_t}{dt}$$

It will turn out to be more convenient to rewrite this expression for how rapidly output Y_t is changing as:

$$\frac{1}{Y_t}\frac{dY_t}{dt} = \alpha\left(\frac{1}{K_t}\frac{dK_t}{dt}\right) + (1-\alpha)\left(\frac{1}{L_t}\frac{dL_t}{dt}\right) + (1-\alpha)\left(\frac{1}{E_t}\frac{dE_t}{dt}\right)$$

and then recognize that the terms inside the second and third parentheses on the right hand side are simply the proportional rate of growth n of the labor force L, and the proportional rate of growth g of the efficiency of labor E.

$$\frac{1}{Y_t}\frac{dY_t}{dt} = \alpha\left(\frac{1}{K_t}\frac{dK_t}{dt}\right) + (1-\alpha)(n+g)$$

The Rate of Change of the Capital Stock K

At any instant, the rate of change of the capital stock K_t is equal to the rate at which new investment is adding to the capital stock minus the rate at which depreciation is subtracting from the capital stock:

$$\frac{dK_t}{dt} = s \times Y_t - \delta \times K_t$$

Thus the proportional rate of change of the capital stock is:

$$\frac{1}{K_t}\frac{dK_t}{dt} = s \times \frac{Y_t}{K_t} - \delta$$

The Rate of Change of the Capital-Output Ratio κ

The capital-output ratio κ_t is equal to the capital stock K_t divided by the level of output Y_t. The calculus rule for the rate of change over time of a quotient like:

$$\kappa_t = \frac{K_t}{Y_t}$$

is:

$$\frac{d\kappa_t}{dt} = \frac{1}{Y_t}\frac{dK_t}{dt} - \frac{K_t}{Y_t^2}\frac{dY_t}{dt}$$

or in more convenient form, dividing both sides by κ_t:

$$\left(\frac{1}{\kappa_t}\frac{d\kappa_t}{dt}\right) = \left(\frac{1}{K_t}\frac{dK_t}{dt}\right) - \left(\frac{1}{Y_t}\frac{dY_t}{dt}\right)$$

Using this, we simply substitute in the expressions for the proportional growth rates of the capital stock and output derived above:

$$\left(\frac{1}{\kappa_t}\frac{d\kappa_t}{dt}\right)=\left(\frac{1}{K_t}\frac{dK_t}{dt}\right)-\alpha\left(\frac{1}{K_t}\frac{dK_t}{dt}\right)-(1-\alpha)(n+g)$$

$$\left(\frac{1}{\kappa_t}\frac{d\kappa_t}{dt}\right)=(1-\alpha)\left[\left(\frac{1}{K_t}\frac{dK_t}{dt}\right)-(n+g)\right]$$

$$\left(\frac{1}{\kappa_t}\frac{d\kappa_t}{dt}\right)=(1-\alpha)\left[\frac{s}{\kappa_t}-\delta-(n+g)\right]$$

And then we multiply by the capital-output ratio to get the rate-of-change of the capital-output ratio by itself on the left hand side:

$$\frac{d\kappa_t}{dt}=(1-\alpha)\left[s-(n+g+\delta)\kappa_t\right]$$

The Path Over Time of the Capital-Output Ratio κ

If we call s/(n+g+δ) by the name κ*, we can rewrite the equation above as:

$$\frac{d\kappa_t}{dt}=-(1-\alpha)(n+g+\delta)\left(\kappa_t-\kappa^*\right)$$

This is a particular *differential equation*: it tells us that the rate at which a variable—in this case κ—changes over time is equal to a constant term (in this case -(1-α)(n+g+δ)) times the gap between the variable's current value and another constant (in this case κ*, equal to (s/(n+g+δ)).

If we know the value of the capital-output ratio κ at any moment—suppose that we know the initial condition, that the capital-output ratio equals some particular value κ_0 when our time measure t=0—then this differential equation has one and only one solution. The solution is:

$$\kappa_t = \kappa^* + (\kappa_0 - \kappa^*) \times e^{-(1-\alpha)(n+g+\delta)t}$$

Where e stands for the base of the natural logarithms, and is a transcendental number the first six significant figures of which are 2.71828.[1]

Let's look at how this equation behaves over time:

- When t=0, then $\kappa_t = \kappa_0$ because the exponent attached to e is equal to zero, and $e^0=1$. This is how it should be: we already knew that when t=0 the value of κ was κ_0.

- When $t=(0.1 \times 1/((1-\alpha)(n+g+\delta)))$, then $\kappa_t = \kappa^* + (\kappa_0-\kappa^*)e^{-0.1} = 0.905(\kappa_0) + 0.095(\kappa^*)$. The capital-output ratio is a weighted average of its initial value and its steady-state equilibrium value. In the amount of time it takes for t to grow to one-tenth the value of the inverse of $(1-\alpha)(n+g+\delta)$, the capital-output ratio has closed 9.5% of the gap between its initial value κ_0 and its steady-state value κ^*.

[1] We had a very general equation, and all of a sudden the very particular number 2.71828… has appeared in its solution. Why this number? Where did it come from? I have never figured that out: it appears that 2.71828 is inscribed in the deep structure not only of this universe but of all possible universes in which there are feedback mechanisms—places where the speed at which something changes depends on its current value.

- When t=1/((1-α)(n+g+δ)), then $\kappa_t = \kappa^* + (\kappa_0 - \kappa^*)e^{-1} = 0.368(\kappa_0) + 0.632(\kappa^*)$. Once again, the capital-output ratio is a weighted average of its initial value and its steady-state equilibrium value. In the amount of time it takes for t to grow to equal the value of the inverse of (1-α)(n+g+δ), the capital-output ratio has closed 63% of the gap between its initial value κ_0 and its steady-state value κ^*.

- t=2/((1-α)(n+g+δ)), then $\kappa_t = \kappa^* + (\kappa_0 - \kappa^*)e^{-2} = 0.135(\kappa_0) + 0.865(\kappa^*)$. In the amount of time it takes for t to grow to twice the value of the inverse of (1-α)(n+g+δ), the capital-output ratio has closed 86% of the gap between its initial value κ_0 and its steady-state value κ^*.

The way to summarize this equation's behavior is that its left hand side variable closes a fraction (1-α)(n+g+δ) of the gap between its current value and its asymptote, its steady-state value, with each tick of the time variable t. Thus the capital-output ratio converges to its steady-state value at a proportional rate of (1-α)(n+g+δ) every year, and the economy converges to its steady-state balanced-growth path.

In the Solow growth model, if the economy is not on its steady-state balanced-growth path, it is heading towards it. The speed at which it converges depends on the value of (1-α)(n+g+δ): the bigger that value, the faster is convergence.

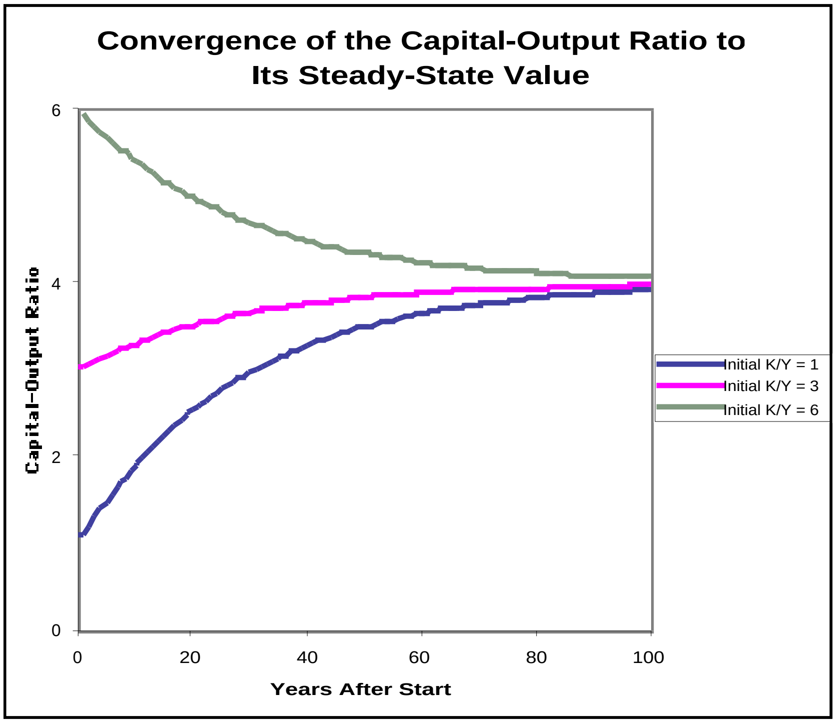

Figure Legend: If the capital-output ratio starts at a value different from steady-state value, it will head towards equilibrium. The figure shows the evolution over time of the capital-output ratio. The underlying parameter values are s=0.28, n=0.02, g=0.015, δ=0.035, α=0.5. The initial starting values of the capital-output ratio are 1, 3, and 6. The steady-state capital-output ratio κ* is 4.