

Lecture Notes: Chapter 4: The Theory of Economic Growth

J. Bradford DeLong
<http://econ161.berkeley.edu/>
delong@econ.berkeley.edu

Background

Ultimately long-run growth is *the* most important aspect of how the economy performs. Material standards of living and levels of economic productivity in the United States today are about four times what they are today in, say, Mexico—and five or so times what they were at the end of the nineteenth century—because of rapid, sustained long-run economic growth. Good and bad policies can accelerate or cripple this growth.

Argentines were richer than Swedes before World War I, but Swedes today have four times the standard of living and the productivity level of Argentines. Almost all of this difference is due to differences in growth policies working through two channels. The first is the impact of policies on the economy's *technology* that multiplies the efficiency of labor. The second is their impact on the economy's *capital intensity*—the stock of machines, equipment, and buildings.

In this growth section, Chapter 5 analyzes the facts of economic growth. Chapter 4 focuses on the theory. Its aim is to build up the growth model that economists use to analyze how much growth is generated by the advance of technology and how much by investment to boost capital intensity on the other.

Better Technology

The bulk of the reason that Americans today are more productive than their predecessors of a century ago is better technology. We now know how to make electric motors, dope semiconductors, transmit signals over fiber optics, fly jet airplanes, machine internal combustion engines, build tall and durable structures out of concrete and steel, record entertainment programs on magnetic tape, make hybrid seeds, fertilize crops with nutrients, organize assembly lines, and a host of other things our predecessors did not know how to do. Better technology leads to a higher *efficiency of labor*--the skills and education of the labor force, the ability of the labor force to handle modern machine

technologies, and the efficiency with which the economy's businesses and markets function.

Capital Intensity

However, a large part is also played by the second factor: *capital intensity*. The more capital the average worker has at his or her disposal to amplify productivity, the more prosperous the economy will be. In turn, there are two principal determinants of capital intensity. The first is the *investment effort* made by the economy: the share of total production--real GDP-- saved and invested to boost the capital stock. The second are the economy's *investment requirements*: how much new investment is needed to simply equip new workers with the standard level of capital, to keep up with new technology, and to replace worn- machines and buildings.

The ratio between the investment effort and the investment requirements of the economy determines the economy's capital intensity. Capital intensity is measured by the economy's capital-output ratio K/Y —the economy's capital stock K divided by its annual real GDP Y —which we will write using a lower-case Greek kappa, κ .

$$\kappa = \frac{K}{Y}$$

The Balanced-Growth Path

In economists' standard *growth model*¹ the type of equilibrium they study is a *balanced-growth equilibrium*. In the balanced-growth equilibrium the capital intensity of the economy—its capital stock divided by its total output—is constant. However, other variables like the capital stock, real GDP, and output per worker are growing.

Economists use the standard model to calculate the balanced-growth path. They then forecast that if the economy is on this path, it will grow along this path. And they forecast that if the economy is not on its balanced growth path, it will head toward that path.

¹ The standard model is called the Solow model, after Nobel Prize-winning M.I.T. economist Robert Solow.

The Steady-State Capital-Output Ratio

What is the economy's balanced-growth path? On the balanced-growth path, the economy's capital-output ratio—which we write κ —is equal to its particular steady-state value κ^* . We calculate this value by taking the share of production that is saved and invested for the future—the economy's saving-investment rate s —and then dividing it by the sum of the depreciation rate at which capital wears out (written δ), the proportional growth rate (written n) of the labor force, and the proportional growth rate (written g) of the efficiency of labor.² In algebra:

$$\kappa^* = \frac{s}{n + g + \delta}$$

Along the balanced-growth path, the level of output per worker Y/L is found by raising the steady-state capital-output ratio κ^* to the power of the growth multiplier (written λ),³ and then multiplying the result by the current efficiency of labor (written E_t). In algebra:

$$\frac{Y_t}{L_t} = \kappa^{*\lambda} \times E_t$$

The steady-state capital-output ratio κ^* is constant (as long as the economy's savings-investment share s , its labor force growth rate n , and its efficiency of labor growth rate g do not change). However, the balanced-growth path level of output per worker is not constant. As time passes, the balanced-growth path level of output per worker rises. Why? Because output per worker Y/L is equal to the *current* efficiency of labor E_t times the steady-state capital-output ratio κ^* raised to the power λ ; and technological progress means that the efficiency of labor E_t grows at a proportional growth rate g .

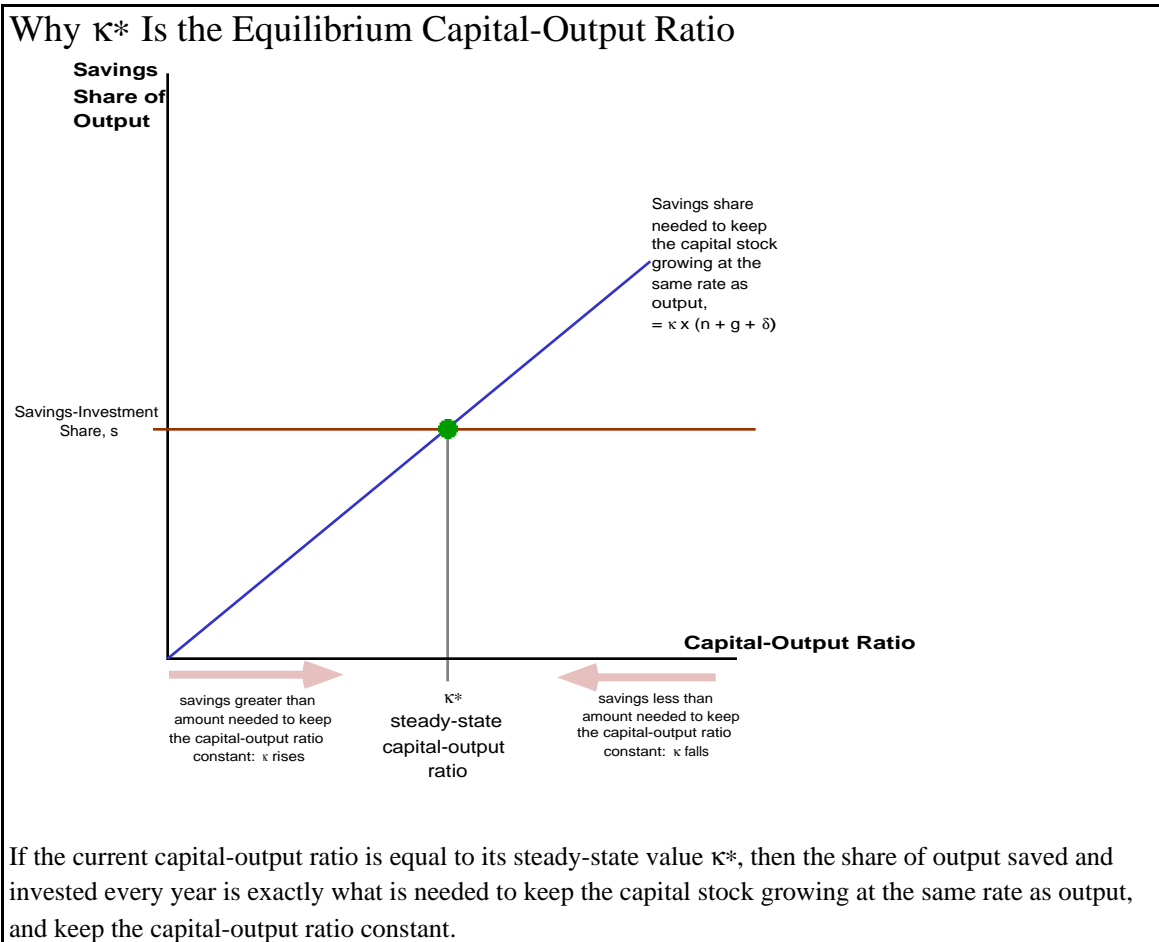
² Recall that we call these last three *investment requirements*.

³ λ , the growth multiplier, is

$$\lambda = \frac{\alpha}{1 - \alpha}$$

where α is the diminishing-returns-to-scale parameter from last chapter's production function $Y/L = (K/L)^\alpha \times E^{1-\alpha}$. It tells by how much (in percentage terms) the economy's output would rise if its capital stock were to grow by one percent.

Is the economy always on its balanced-growth path? No. But if the economy is not on it, it is heading towards it.



If the capital-output ratio κ is below κ^* , the share of output invested each year (equal to s) is greater than needed to keep the capital stock growing as fast as output (equal to $\kappa(n + g + \delta)$). The capital-output ratio rises. If the capital-output ratio is above κ^* , the share invested each year (equal to s) is less than needed to keep the capital stock growing as fast as output (equal to $\kappa(n + g + \delta)$). The capital-output ratio falls. The economy closes some of the gap between its current position and its steady-state growth path.

The Determinants of the Balanced-Growth Path

Thus the steady-state balanced growth path depends on five factors:

- the economy's savings-investment rate, the share of output used to buy investment goods to boost the capital stock (written s)
- the growth rate of the efficiency of labor (written g)
- the depreciation rate—the proportion of the existing capital stock K that wears out or becomes obsolete every year (written δ)
- the economy's labor force growth rate (written n)
- the economy's growth multiplier (written λ , equal to $\alpha/(1-\alpha)$, where α comes from the production function)
- the current efficiency of labor—a measure of the economy's ability to use technology, where “technology” is defined in the broadest possible sense to include work organization, incentives, and all other factors that affect the ability of the economy to use resources to produce goods and services. (written E_t).

Factors (1) through (4) determine the steady-state capital-output ratio κ^* , which is then raised to the λ power (factor (5)), and the result is then multiplied by the current efficiency of labor E_t (factor (6)).

Forecasting the Economy's Destiny

The standard Solow growth model makes forecasting an economy's long-run growth destiny simple:

- Calculate the steady-state capital-output ratio, $\kappa^* = s/(n+g+\delta)$, equal to the savings share divided by the investment requirements.
- Amplify the steady-state capital-output ratio κ^* by raising it to the power of the growth multiplier $\lambda = (\alpha/(1-\alpha))$, where α is the production function's diminishing-returns-to-scale parameter.
- Multiply the result by the current efficiency of labor E_t .

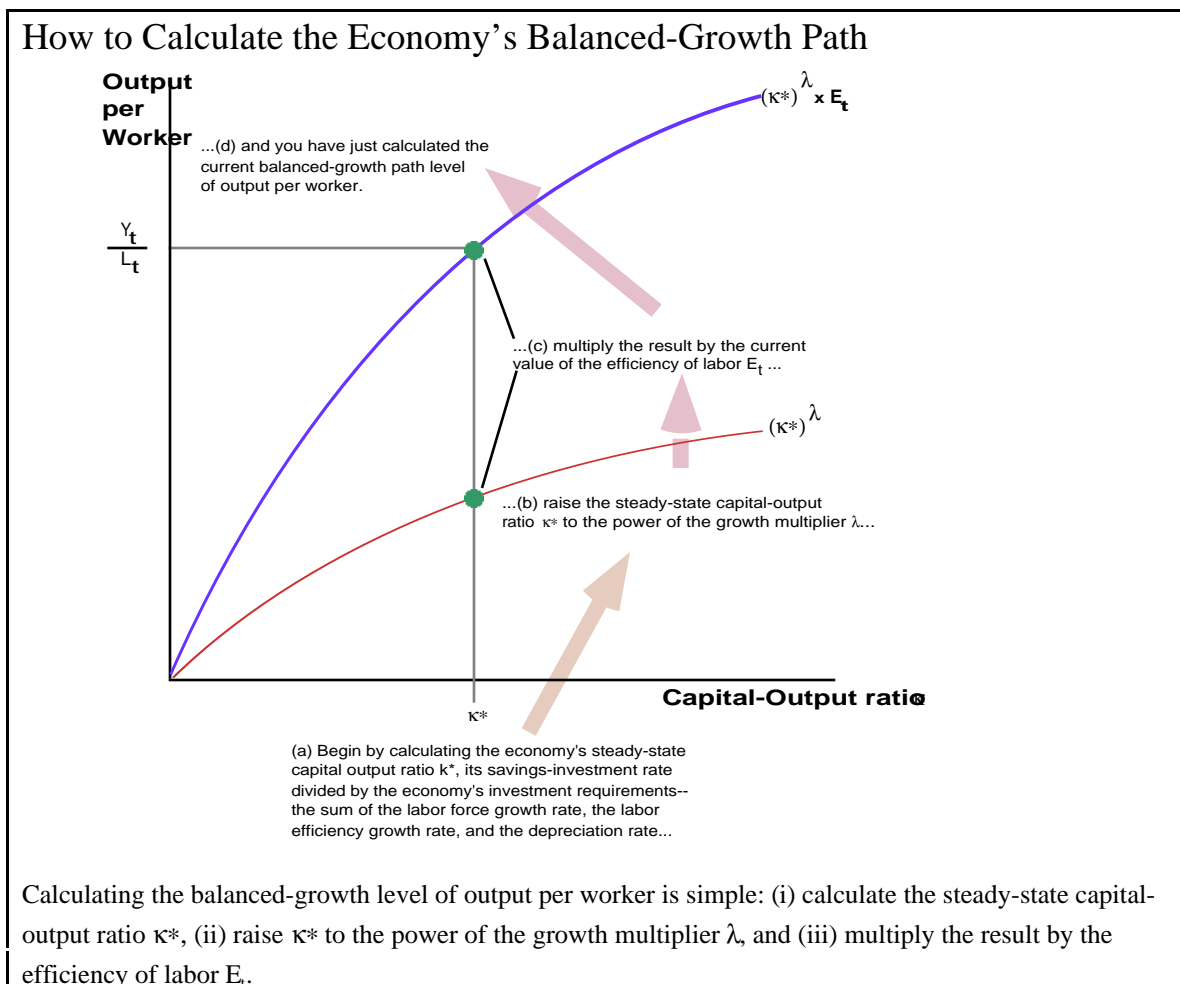
You have just calculated output per worker on the economy's *balanced-growth path*. If you just want to understand the present, you are done. If you want to also forecast the future, then:

- Forecast that balanced-growth output per worker will grow at the same proportional rate g as labor efficiency.

If the economy is on its balanced-growth path, you are done. But if the economy is not currently on its balanced-growth path, then:

- Forecast that the economy is heading for its balanced-growth path.
- Forecast that the economy will grow along its balanced-growth path after it has converged to it.

The growth model makes forecasts of the long-run destiny of the economy straightforward, and provides an easy way to analyze how the factors making for (a) higher capital intensity and (b) better technology and labor efficiency determine output per worker.



What happens if there is a change in economic policy or in the economic environment—if something happens to change s , the savings rate; n , the labor force growth rate; g , the efficiency of labor growth rate; or δ , the depreciation rate? If any of

these change, the economy's steady-state balanced-growth path changes as well. So calculate the new steady-state balanced-growth path, and then forecast that the economy will converge to this new balanced-growth path.

Why, and how, does this growth model work? Why is there a steady-state growth path? Why do these calculations above tell us what it is? To understand these issues, we need to back up and dig a little deeper. To explain them is the business of Chapter 4, to which we now turn.

The Standard Growth Model

Economists begin to analyze long-run growth by building a simple, standard model of economic growth—a *growth model*. This standard model is also called the Solow model, after Nobel Prize-winning M.I.T. economist Robert Solow. The second thing economists do is to use the model to look for an *equilibrium*--a point of balance, a condition of rest, a state of the system toward which the model will converge over time. Once you have found the equilibrium position toward which the economy tends to move, you can use it to understand how the model will behave. If you have built the right model, this will tell you in broad strokes how the economy will behave.

In economic growth economists look for the *steady-state balanced-growth equilibrium*. In a steady-state balanced-growth equilibrium the capital intensity of the economy is stable. The economy's capital stock and its level of real GDP are growing at the same proportional rate. And the capital-output ratio--the ratio of the economy's capital stock to annual real GDP--is constant.

The Production Function

The first component of the model is a *behavioral relationship* called the *production function*. This behavioral relationship tells us how the productive resources of the economy—the labor force, the capital stock, and the level of technology that determines the efficiency of labor—can be used to produce and determine the level of output in the economy. The total volume of production of the goods and services that consumers, investing businesses, and the government wish for is limited by the available resources. The production function tells us how available resources limit production.

Tell the production function what resources the economy has available, and it will tell you how much the economy can produce. We will use the so-called Cobb-Douglas

production function, a functional form that economists use because it makes many kinds of calculations relatively simple. The Cobb-Douglas production function states:

$$(Y/L) = (K/L)^\alpha \times (E)^{1-\alpha}$$

The economy's level of output per worker (Y/L) is equal to the capital stock per worker K/L raised to the exponential power of some number α , and then multiplied by the current efficiency of labor E raised to the exponential power $(1 - \alpha)$.

The efficiency of labor E and the number α are *parameters* of the model. The parameter α is always a number between zero and one. The best way to think of it as the parameter that governs how fast diminishing returns to investment set in. A level of α near zero means that the extra amount of output made possible by each additional unit of capital declines very quickly as the capital stock increases.

By contrast, a level of α near one means that the next additional unit of capital makes possible almost as large an increase in output as the last additional unit of capital, as Figure 4.2 shows. When α equals one, output is proportional to capital: double the stock of capital per worker, and you double output per worker as well. When the parameter α is near to but less than one, diminishing returns to capital accumulation do set in, but they do not set in rapidly or steeply. And as α varies from a high number near one to a low number near zero, the force of diminishing returns gets stronger.

The other parameter E tells us the current level of the efficiency of labor. A higher level of E means that more output per worker can be produced for each possible value of the capital stock per worker. A lower value of E means that the economy is very unproductive: not even huge amounts of capital per worker will boost output per worker to achieve what we would think of as prosperity. Box 4.1 illustrates how to use the production function once you know its form and parameters--how to calculate output per worker once you know the capital stock per worker.

The Cobb-Douglas production function is "flexible" in the sense that it can be tuned to fit any of a wide variety of different economic situations. Figure 4.3 shows a small part of the flexibility of the Cobb-Douglas production function. Is the level of productivity high? The Cobb-Douglas function can fit with a high initial level of the efficiency of labor E . Does the economy rapidly hit a wall as capital accumulation proceeds and find that all the investment in the world is doing little to raise the level of production? Then the Cobb-Douglas function can fit with a low level--near zero--of the diminishing-returns-to-capital

parameter α . Is the speed with which diminishing-returns-to-investment sets in moderate? Then pick a moderate value of α , and the Cobb-Douglas function will once again fit.

No economist believes that there is, buried somewhere in the earth, a big machine that forces the level of output per worker to behave exactly as calculated by the algebraic production function above. Instead, economists think that the Cobb-Douglas production function above is a simple and useful approximation.

The true process that does determine the level of output per worker is an immensely complicated one: everyone in the economy is part of it. And it is too complicated to work with. Writing down the Cobb-Douglas production function is a breathtakingly large leap of abstraction. Yet it is a useful leap, for this approximation is good enough that using it to analyze the economy will get us to approximately correct conclusions.

The Rest of the Growth Model

The rest of the growth model is straightforward. First comes the need to keep track of the quantities of the model over time. Do so by attaching to each variable--like the capital stock or the efficiency of labor or output per worker or the labor force--a little subscript telling what year it applies to. Thus K_{1999} will be the capital stock in year 1999. If we want to refer to the efficiency of labor in the current year (but don't much care what the current year is), we will use a t (for "time") as a placeholder to stand in for the numerical value of the current year. Thus we write: E_t . And if we want to refer to the efficiency of labor in the year after the current year, we will write: E_{t+1} .

We assume—once again making a simplifying leap of abstraction--that the labor force L of the economy is growing at a constant proportional rate given by the value of a parameter n . Note that n does not have to be the same across countries, and can shift over time in any one country). Thus between this year and the next the labor force will grow so that:

$$L_{t+1} = (1 + n) \times L_t$$

Assume, also, that the efficiency of labor E is growing at a constant proportional rate given by a parameter g . (Note that g does not have to be the same across countries, and can shift over time in any one country.) Thus between this year and the next year:

$$E_{t+1} = (1 + g) \times E_t$$

Last, assume that a constant proportional share, equal to a parameter s , of real GDP is saved each year and invested. These gross investments add to the capital stock, so a higher amount of savings and investment means faster growth for the capital stock.

But the capital stock does not grow by the full amount of *gross* investment. A fraction δ (the Greek letter lower-case delta, for depreciation) of the capital stock wears out or is scrapped each period. Thus the actual relationship between the capital stock now and the capital stock next year is:

$$K_{t+1} = K_t + (s \times Y_t) - (\delta \times K_t)$$

The level of the capital stock next year will be equal to the capital stock this year, plus the savings rate s times this year's level of real GDP, minus the depreciation rate δ times this year's capital stock.

That is all there is to the growth model: three assumptions about rates of population growth, increases in the efficiency of labor, and investment, plus one additional equation to describe how the capital stock grows over time. Those plus the production function make up the growth model. It is simple. But understanding the processes of economic growth that the model generates is more complicated.

Understanding the Growth Model

Economists' first instinct when analyzing any model is to look for a point of *equilibrium*. In the study of long-run growth, however, the key economic quantities are never stable. They are growing over time. The efficiency of labor is growing, the level of output per worker is growing, the capital stock is growing, the labor force is growing. How, then, can we talk about a point of equilibrium where things are stable if everything is growing?

The answer is to look for an equilibrium in which everything is growing together, at the same proportional rate. Such an equilibrium is one of *steady-state balanced growth*. If everything is growing together, then the relationships between key quantities in the economy are stable. And it makes this chapter easier if we focus on one key ratio: the capital-output ratio. Thus our point of equilibrium will be one in which the capital-output ratio is constant over time, and toward which the capital-output ratio will converge if it should find itself out of equilibrium.

How Fast Is the Economy Growing?

So how fast are the key quantities in the economy growing? Determining how fast the quantities in the economy are growing is straightforward if we remember our three mathematical rules:

- The proportional growth rate of a product -- $P \times Q$, say--is equal to the sum of the proportional growth rates of the factors, is equal to the growth rate of P plus the growth rate of Q .
- The proportional growth rate of a quotient-- E/Q , say--is equal to the difference of the proportional growth rates of the dividend (E) and the divisor (Q).
- The proportional growth rate of a quantity raised to a exponent-- Q^y , say--is equal to the exponent (y) times the growth rate of the quantity (Q).

The Growth of Capital per Worker

Begin with capital per worker. To save on our breath and reduce the length of equations, let's use the expression $g(k_t)$ to stand for the proportional growth rate of capital per worker. The proportional growth rate is simply what output per worker will be next year minus what output per worker is this year, all divided by what output per worker is this year:

$$g(k_t) = \frac{(K_{t+1}/L_{t+1}) - (K_t/L_t)}{(K_t/L_t)}$$

Capital-per-worker is a quotient: it is the capital stock divided by the labor force. Thus the proportional growth rate of capital-per-worker is the growth rate of the capital stock minus the growth rate of the labor force.

The growth rate of the labor force is simply the parameter n . That's what the parameter n is. The growth rate of the capital stock is a bit harder to calculate. We know that it is:

$$\frac{K_{t+1} - K_t}{K_t}$$

And we know that we can write next year's capital stock as equal to this year's capital stock, plus gross investment, minus depreciation:

$$K_{t+1} = (K_t + (s \times Y_t) - (\delta \times K_t))$$

If we substitute in for next year's capital stock, and rearrange:

$$\frac{(K_t + (s \times Y_t) - (\delta \times K_t)) - K_t}{K_t} = \frac{s \times Y_t}{K_t} - \delta \frac{K_t}{K_t} = \frac{s \times Y_t}{K_t} - \delta$$

Then we see that the proportional growth rate of capital per worker is:

$$g(k_t) = \frac{s}{(K_t / Y_t)} - \delta - n$$

To make our equations look simpler, let's give the capital-output ratio K/Y a special symbol: κ --a little k with a short stem (actually the Greek letter kappa)--and write that the proportional growth rate of capital per worker is:

$$g(k_t) = s / \kappa_t - \delta - n$$

This says that the growth rate of capital-per-worker is equal to the savings share of GDP (s) divided by the capital-output ratio (κ), minus the depreciation rate (δ), minus the labor force growth rate (n). Box 4.3 goes through example calculations of what the growth rate of capital-per-worker is for sample parameter values. A higher rate of labor force growth will reduce the rate of growth of capital per worker: more workers means the available capital has to be divided up more ways. A higher rate of depreciation will reduce the rate of growth of capital per worker: more capital will rust away. And a higher capital-output ratio will reduce the proportional growth rate of capital per worker: the higher the capital-output ratio, the smaller is investment relative to the capital stock.

The Growth of Output per Worker

Our Cobb-Douglas form of the production function tells us that the level of output per worker is:

$$(Y_t / L_t) = (K_t / L_t)^\alpha \times (E_t)^{1-\alpha}$$

Output per worker is the product of two terms, each of which is a quantity raised to a exponential power. So using our mathematical rules of thumb the proportional growth rate of output per worker--call it $g(y)$ to once again save on space--will be, as Figure 4.8 shows:

- α times the proportional growth rate of capital per worker,
- plus $(1 - \alpha)$ times the rate of growth of the efficiency of labor.

The rate of growth of the efficiency of labor is simply g . And the previous section calculated the growth rate of capital per worker $g(k)$: $s/\kappa_t - \delta - n$.

So simply plug these expressions in:

$$g(y_t) = [\alpha \times \{s/\kappa_t - \delta - n\}] + [(1 - \alpha) \times g]$$

And simplify a bit by rearranging terms:

$$g(y_t) = g + [\alpha \times \{s/\kappa_t - (n + g + \delta)\}]$$

The Growth of the Capital-Output Ratio

Now consider the capital-output ratio κ_t . It will be the key ratio that we will focus on-- and our equilibrium will be when it is stable and constant. The capital-output ratio is equal to capital per worker divided by output per worker. So its proportional growth rate is the difference between their growth rates:

$$g(\kappa_t) = g(k_t) - g(y_t) = \{s/\kappa_t - \delta - n\} - \{g + \alpha \times \{s/\kappa_t - (n + g + \delta)\}\}$$

Which simplifies to:

$$g(\kappa_t) = (1 - \alpha) \times \{s/\kappa_t - (n + g + \delta)\}$$

Thus the growth rate of the capital-output ratio depends on the balance between the *investment requirements*— $(n+g+\delta)$ --and the *investment effort*— s --made in the economy.

From the growth rate of the capital-output ratio:

$$g(\kappa_t) = (1 - \alpha) \times \{s/\kappa_t - (n + g + \delta)\}$$

We can see that whenever the capital-output ratio κ_t is greater than $s/(n+g+\delta)$, the growth rate of the capital-output ratio will be negative. Output per worker will be growing faster than capital per worker. And the capital-output ratio will be shrinking. By contrast, we can also see that whenever the capital-output ratio κ_t is *less* than $s/(n+g+\delta)$, the capital-output ratio will be growing.

What happens when the capital-output ratio κ_t is equal to $s/(n+g+\delta)$? Then the growth rate of the capital-output ratio will be zero. It will be stable, neither growing nor shrinking. If the capital-output ratio is at that value, it will stay there. If the capital-output ratio is away from that value, it will head toward there. No matter where the capital-output ratio κ_t starts, it will head for--converge to--home in on-- its steady-state balanced-growth value of $s/(n+g+\delta)$ (see Figure 4.11).

Thus the value $s/(n+g+\delta)$ is the *equilibrium level* of the capital-output ratio. It is a point at which the economy tends to balance, and to which the economy converges. The requirement that the capital-output ratio equal this equilibrium level becomes our equilibrium condition for balanced economic growth.

And to make our future equations even simpler, give the quantity $s/(n+g+\delta)$ that is the equilibrium value of the capital-output ratio a special symbol: κ^* :

$$\kappa^* = \frac{s}{n + g + \delta}$$

Other Quantities

When the capital-output ratio κ_t is at its steady state value of:

$$\kappa^* = s/(n+g+\delta),$$

the proportional growth rates of capital per worker and output per worker are stable too.

Output per worker is then growing at a proportional rate g :

$$g(y_t) = g$$

The capital stock per worker is then growing at the same proportional rate g :

$$g(k_t) = g$$

The total economy-wide capital stock is then growing at the proportional rate $n+g$: the growth rate of capital per worker plus the growth rate of the labor force. Real GDP is then also growing at rate $n+g$: the growth rate of output per worker plus the labor force growth rate.

The Level of Output per Worker On the Steady State Growth Path

When the capital-output ratio is at its steady-state balanced-growth equilibrium value κ^* , we say that the economy is on its steady-state growth path. What is the level of output per worker if the economy is on its steady-state growth path?

If we define:

$$\lambda = \frac{\alpha}{1 - \alpha}$$

and call λ the *growth multiplier*, then output per worker along the steady-state growth path is equal to the steady-state capital-output ratio raised to the growth multiplier, times the current level of the efficiency of labor.:

$$\left(\frac{Y_t}{L_t} \right)_{ss} = \kappa^{*\lambda} \times E_t$$

Thus calculating output per worker when the economy is on its steady-state growth path is a simple three-step procedure:

- First, calculate the steady-state capital-output ratio, $\kappa^* = s / (n + g + \delta)$, the savings rate divided by the sum of the population growth rate, the efficiency of labor growth rate, and the depreciation rate.
- Second, amplify the steady-state capital-output ratio κ^* by the growth multiplier. Raise it to the $\lambda = (\alpha / (1 - \alpha))$ power, where α is the diminishing-returns-to-capital parameter.
- Third, multiply the result by the current value of the efficiency of labor E_t , which can be easily calculated because the efficiency of labor is growing at the constant proportional rate g .

And the fact that an economy converges to its steady-state growth path makes analyzing the long-run growth of an economy relatively easily as well:

- First calculate the steady-state growth path, shown in Figure 4.13.
- From the steady-state growth path, forecast the future of the economy: If the economy is on its steady-state growth path today, it will stay on its steady-state growth path in the future (unless some of the parameters-- n , g , δ , s , and α —shift).
- If the economy is not on its steady-state growth path today, it is heading for its steady-state growth path and will get there soon.

Thus long-run economic forecasting becomes simple.

How Fast Does the Economy Head For Its Steady-State Growth Path?

The growth rate of the capital-output ratio will be approximately equal to a fraction $(1-\alpha) \times (n+g+\delta)$ of the gap between the steady-state and its current level.

For example, if $(1-\alpha) \times (n+g+\delta)$ is equal to 0.04, the capital-output ratio will close approximately 4 percent of the gap between its current level and its steady-state value in a year. If $(1-\alpha) \times (n+g+\delta)$ is equal to 0.07, the capital-output ratio closes 7 percent of the gap between its current level and its steady-state value in a year. A variable closing 4 percent of the gap each year between its current and its steady-state value will move halfway to its steady-state value in 18 years. A variable closing 7 percent of the gap each year between its current and its steady-state value will move halfway to its steady-state value in 10 years.