Linear First Order Differential Equation

Separable Equations

We are looking for y[x] that satisfies the equation; so we "separate" the x and y[x] to different sides and take integral of both sides:

Example: $\frac{y'[x]}{2x} = 1 \Rightarrow y'[x] = 2x \Rightarrow \int y'[x] \, dy = \int 2x \, dx \Rightarrow y[x] = x^2 + c$

Example: $\frac{y'[x]}{2x\, y[x]} = 1 - 3x \Rightarrow \frac{y'[x]}{y[x]} = 2x(1-3x) \Rightarrow \int \frac{y'[x]}{y[x]} \, dy = \int 2x(1-3x) \, dx \Rightarrow \text{Ln}(y[x]) = x^2 - 2x^3 + c \Rightarrow$

$y[x] = e^{x^2 - 2x^3 + c} = C_0 \, e^{x^2 - 2x^3}$

Solution using simple technique

General Form: **$x'[t] + P[t]\, x[t] = Q[t]$** (1)

Let: $\rho[t] = e^{\int P[t]\, dt}$; multiply both sides of (1) by $\rho[t] \Rightarrow$

$e^{\int P[t]\, dt}\, x'[t] + P[t]\, e^{\int P[t]\, dt}\, x[t] = Q[t]\, e^{\int P[t]\, dt}$ (2)

Recognize the left side as $\left(x[t]\, e^{\int P[t]\, dt}\right)'$ so:

$\left(x[t]\, e^{\int P[t]\, dt}\right)' = Q[t]\, e^{\int P[t]\, dt}$ (3)

now take integral of both sides to get:

$x[t]\, e^{\int P[t]\, dt} = \int Q[t]\, e^{\int P[t]\, dt}\, dt + C \quad \Rightarrow$

$$\boxed{x[t] = \frac{\int Q[t]\, e^{\int P[t]\, dt}\, dt + C}{e^{\int P[t]\, dt}} = \frac{\int Q[t]\, \rho[t]\, dt + C}{\rho[t]}}$$

Example:

$x'[t] - 3x[t] = e^{2t}$; $y[0] = 3$

$P[t] = -3$; $Q[t] = e^{2t}$; $\rho[t] = e^{\int P[t]\, dt} = e^{-3t}$

$$x[t] = \frac{\int Q[t]\, \rho[t]\, dt + C}{\rho[t]} = \frac{\int e^{2t}\, e^{-3t}\, dt + C}{e^{-3t}} = \frac{-e^{-t} + C}{e^{-3t}}$$

$$x[0] = 3 \Rightarrow C = 4 \Rightarrow x[t] = \frac{-e^{-t} + 4}{e^{-3t}} = -e^{2t} + 4\, e^{3t}$$

■ Solution Using Laplace Transform

Definition of Laplace Transform: $L\{f[t]\} = \int_0^\infty e^{-st} f[t]\, dt = F[s]$;

also the inverse Laplace Transform exists: $L^{-1}\{F[s]\} = f[t]$

Examples of Laplace Transforms:

$f[t] = 1 \Rightarrow F[s] = \frac{1}{s}$ $(s > 0)$

$f[t] = t \Rightarrow F[s] = \frac{1}{s^2}$ $(s > 0)$

$f[t] = t^n \Rightarrow F[s] = \frac{n!}{s^{n+1}}$ $(s > 0)$

$f[t] = t^a\ (a > -1) \Rightarrow F[s] = \frac{\Gamma[a+1]}{s^{a+1}}$ $(s > 0)$ (where $\Gamma[x] = \int_0^\infty e^{-t} t^{x-1}\, dt$ is the famous Gamma function)

$f[t] = e^{at} \Rightarrow F[s] = \frac{1}{s-a}$ $(s > 0)$

also (lucky for us) Laplace transforms are Linear Operations; namely for $a, b \in R$:

$L\{a f[t] + b g[t]\} = a L\{f[t]\} + b L\{g[t]\}$

Last thing to know here is the transform of derivatives (with some conditions):

$L\{f'[t]\} = s L\{f[t]\} - f[0]$

In this case we can solve a linear differential equation:

$f'[t] - 3 f[t] = e^{2t};\ f[0] = 3$

Step 1: Do Laplace transform of both sides:

$s L\{f[t]\} - f[0] - 3 L\{f[t]\} = \frac{1}{s-2} \Rightarrow$

Step 2: taking into account that the intial condition $f[0] = 3$; we solve for $L\{f[t]\}$:

$L\{f[t]\}(s-3) = \frac{1}{s-2} + 3 \Rightarrow$

$L\{f[t]\} = \frac{\frac{1}{s-2} + 3}{s-3} = \frac{4}{s-3} - \frac{1}{s-2} \Rightarrow$

Step 3: To know $f[t]$ we do inverse Laplace transform:

$L^{-1}\{F[s]\} = L^{-1}\left\{\frac{4}{s-3} - \frac{1}{s-2}\right\} = f[t] = 4 e^{3t} - e^{2t}$ which is exactly what we got in the previous example.