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J.P.'s "Rough Guide" to
1^{st} Order Autonomous Differential Equations

Written for ECON 101b – Fall 1999

Define $\dot{x}(t) = \frac{dx(t)}{dt}$

Here is an example of a differential equation:

$$\dot{x}(t) = ax(t) + b(t)$$

This is a *first order* differential equation because it only involves the first derivative of $x(t)$ with respect to time. We say it's *autonomous* because, it does not involve time is not an explicit and separate argument of the function. It consists of a homogenous part, $ax(t)$, and a non-homogenous part, $b(t)$.

Solution to the homogenous part, *i.e.* to $\dot{x}(t) = ax(t)$

At a moment *s*,

$$\frac{dx(s)}{ds} = ax(s) \tag{1}$$

Rearranging terms we can rewrite (1) as:

$$\frac{1}{x(s)}dx(s) = ads \tag{2}$$

The "trick" consists now in integrating both sides from *s=0* to *t*.

$$\int_{s=0}^{t} \frac{1}{x(s)}dx(s) = \int_{s=0}^{t} ads \tag{3}$$

Using the Fundamental Theorem of Calculus, we obtain:

$$\ln(x(t)) - \ln(x(0)) = a.t - a.0 \Rightarrow \ln(x(t)) = \ln(x(0)) + a.t \tag{4}$$

Taking the exponential function on both sides, we have:

$$\exp(\ln(x(t))) = \exp(\ln(x(0)) + a.t) \tag{5}$$

Or, $\boxed{x(t)=x(0)\exp(at)}$ ☺ (6)

That last equation is what we're looking for: an expression that directly tells you what x(t) is equal to as a function of time.

Solution to the complete equation with a constant non-homogenous term, *i.e.* to $\dot{x}(t)=ax(t)+b$

At a moment *s*,

$$\frac{dx(s)}{ds}=ax(s)+b \tag{1'}$$

Rearranging terms we can rewrite (1') as:

$$dx(s)=ax(s)ds+bds \Leftrightarrow dx(s)-ax(s)ds=bds \tag{2'}$$

Now multiply both sides of (2') by $\exp(-as)$:

$$\exp(-as)dx(s)-\exp(-as)ax(s)ds=\exp(-as)bds \tag{3'}$$

And recognize that $d[x(s)\exp(-as)]=\exp(-as)dx(s)-\exp(-as)ax(s)ds$, so,

$$d[x(s)\exp(-as)]=\exp(-as)bds \tag{4'}$$

Here also we need to integrate both of (4') sides from *s=0* to *t*.

$$\int_{s=0}^{t} d[x(s)\exp(-as)]=\int_{s=0}^{t}\exp(-as)bds \tag{5'}$$

Using the Fundamental Theorem of Calculus, we obtain:

$$x(t)\exp(-at)-x(0)\exp(-a.0)=-\frac{b}{a}\left(\exp(-at)-\exp(-a.0)\right)\Leftrightarrow \tag{6'}$$

$$x(t)\exp(-at)-x(0)=\frac{b}{a}\left(1-\exp(-at)\right)$$

After rearranging terms:

$$\boxed{x(t)=\left(x(0)+\frac{b}{a}\right)\exp(at)-\frac{b}{a}} \text{ ☺} \tag{7'}$$

That last equation is again what we're looking for: an expression that directly tells you what x(t) is equal to as a function of time.

Solution to the complete equation with a time-varying non-homogenous term, *i.e.* to $\dot{x}(t) = ax(t) + b(t)$

At a moment *s*,

$$\frac{dx(s)}{ds} = ax(s) + b(s) \tag{1''}$$

Rearranging terms we can rewrite (1'') as:

$$dx(s) = ax(s)ds + b(s)ds \Leftrightarrow dx(s) - ax(s)ds = b(s)ds \tag{2''}$$

Now multiply both sides of (2'') by $\exp(-as)$:

$$\exp(-as)dx(s) - \exp(-as)ax(s)ds = \exp(-as)b(s)ds \tag{3''}$$

And recognize that $d[x(s)\exp(-as)] = \exp(-as)dx(s) - \exp(-as)ax(s)ds$, so,

$$d[x(s)\exp(-as)] = \exp(-as)b(s)ds \tag{4''}$$

Here also we need to integrate both of (4'') sides from *s=0* to *t*.

$$\int_{s=0}^{t} d[x(s)\exp(-as)] = \int_{s=0}^{t} \exp(-as)b(s)ds \tag{5''}$$

Using again the Fundamental Theorem of Calculus, we obtain:

$$x(t)\exp(-at) - x(0)\exp(-a.0) = \int_{s=0}^{t} \exp(-as)b(s)ds \Leftrightarrow \tag{6''}$$

$$x(t)\exp(-at) - x(0) = \int_{s=0}^{t} \exp(-as)b(s)ds$$

After rearranging terms:

$$x(t) = x(0)\exp(at) + \exp(at)\int_{s=0}^{t} \exp(-as)b(s)ds \tag{7''}$$

Note that $\exp(at)$ does not depend upon the variable of integration, *s*, and can therefore be moved inside the integral:

$$\boxed{x(t) = x(0)\exp(at) + \int_{s=0}^{t} \exp[a(t-s)]b(s)ds} \quad ☺ \qquad (7'')$$

That last equation is once more what we’re looking for: an expression that directly tells you what x(t) is equal to as a function of time.

Reference

Michael W. Klein, *Mathematical Methods for Economists,* Addison Wesley: Reading, Massachusetts, 1998: pp. 443-481