Econ 101b - Answer Key to Problem Set 2

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Question 1

Consider the production function:

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{0.5} E^{1-0.5}$$

a. Suppose E=1, L=100, and K=64; what is output per worker Y/L?

$$\left(\frac{64}{100}\right)^{0.5} 1^{1-0.5}$$

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0.8
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b. Suppose E=3, L=196, and K=49; what is output per worker Y/L?

$$\left(\frac{49}{196}\right)^{0.5} 3^{1-0.5} = \frac{\sqrt{3}}{2}$$

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0.866025
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The wisdom to be gained from a and b is that differences in labor efficiency (i.e. technological progress) can easily overweight differences in capital-labor ratios.

c. If both capital K and labor L double, what happens to total output Y? (Not output per worker Y/L, but total output.)

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{0.5} E^{1-0.5} \Rightarrow Y = \left(\frac{K}{L}\right)^{0.5} L E^{1-0.5} \Rightarrow Y = K^{0.5} (LE)^{1-0.5}$$

$$(2K)^{0.5} (2LE)^{1-0.5} = 2^{0.5+1-0.5} K^{0.5} (LE)^{1-0.5} = 2K^{0.5} (LE)^{1-0.5} = 2Y$$

Thus, if you double both inputs, you double output. This production function thus has *constant returns to scale.*

d. Holding E=1, suppose that capital per worker increases from 2 to 4 and then from 4 to 6. What happens to output per worker?

$$(2)^{0.5}\, 1^{1-0.5}$$

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1.41421
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$$(4)^{0.5}\, 1^{1-0.5}$$

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2.
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Thus output per worker increased by:

$$(4)^{0.5}\, 1^{1-0.5} - (2)^{0.5}\, 1^{1-0.5}$$

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0.585786
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Now,

$$(6)^{0.5}\, 1^{1-0.5}$$

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2.44949
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Thus output per worker increased by:

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(6)^0.5 1^(1-0.5) - (4)^0.5 1^(1-0.5)
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0.44949
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Thus by less than when the capital-outputwent up from 4 to 6.

Our productionfunctionis actually:

$$y = \frac{Y}{L} = \left(\frac{K}{L}\right)^{0.5} = k^{0.5}$$

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f[k] := k^0.5
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Plot[f[k], {k, 0, 20}]
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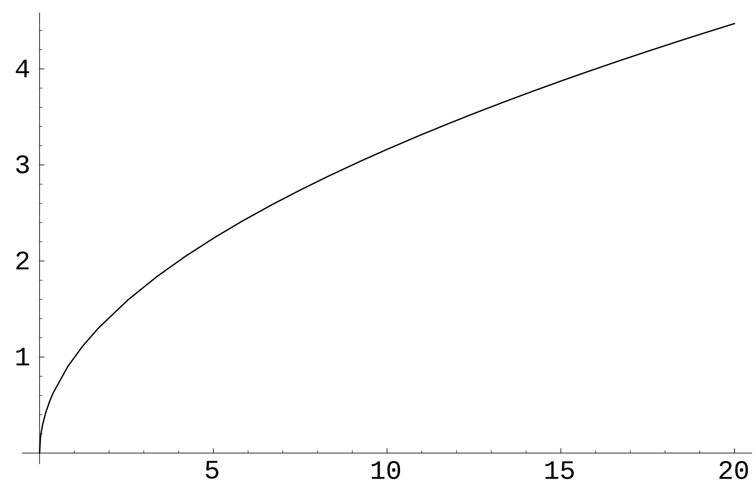

This is how our production function looks like; it obviously displays diminishing returns to capital and thus to the capital-laborratio. In other words, increasing the capital-laborratio improves output per worker by progressively decreasing amounts.

Question 2.

Consider an economy in which the depreciation rate is 3% per year, the rate of population increase is 1% per year, the rate of technological progress is 1% per year, and the private savings rate is 16% of GDP. Suppose that the government increases its budget deficit--which had been at 1% of GDP for a long time--to 3.5% of GDP and keeps it there indefinitely.

a. What is the effect of this shift on the economy’s steady-state capital-output ratio?

Recall that $S = S_p + S_g$, i.e. national saving is made of private saving and public saving.

Also recall that S=-(G-T)=-(Budget Deficit)

Thus s = average propensity to save = $\frac{S}{Y} = \frac{S_p + S_g}{Y} = \frac{S_p}{Y} + \frac{S_g}{Y} = s_p + s_g$

That is the national propensity to save, s, is the sum of the private savings rate, s_p, and the budget deficit as a percentage of output, s_g.

Originally, $\left(\frac{K}{Y}\right)_{SS,old} = \frac{s}{n+g+\delta} = \frac{s_p+s_g}{n+g+\delta}$

$$\frac{0.16 - 0.01}{0.01 + 0.01 + 0.03}$$

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3.
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When the government increases its budget deficit--which had been at 1% of GDP for a long time--to 3.5% of GDP and keeps it there indefinitely, $\left(\frac{K}{Y}\right)_{SS,new} = \frac{s_p+s_g}{n+g+\delta} =$

$$\frac{0.16 - 0.035}{0.01 + 0.01 + 0.03}$$

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2.5
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b. What is the effect of this shift on the economy’s steady state growth path for output per worker?

$$\frac{\left(\frac{Y}{L}\right)_{t,new}}{\left(\frac{Y}{L}\right)_{t,old}} = \frac{\left(\frac{s_p+s_{g,new}}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} E_t}{\left(\frac{s_p+s_{g,old}}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} E_t} = \left(\frac{s_p + s_{g,new}}{s_p + s_{g,old}}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{0.16 - 0.035}{0.16 - 0.01}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{5}{6}\right)^{\frac{\alpha}{1-\alpha}}$$

Thus output per worker at any point of time will be higher than when the budget deficit as a percentage of GDP was equal to 1% since we only consider values of $\alpha \in [0,1]$

c. Suppose that your forecast of output per worker 20 years in the future had been $100,000. What is your new forecast of output per worker twenty years hence?

$$\left(\frac{Y}{L}\right)_{t+20,new} = \frac{\left(\frac{Y}{L}\right)_{t+20,new}}{\left(\frac{Y}{L}\right)_{t+20,old}} \left(\frac{Y}{L}\right)_{t+20,old} = \left(\frac{s_p + s_{g,new}}{s_p + s_{g,old}}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{Y}{L}\right)_{t+20,old} = \left(\frac{5}{6}\right)^{\frac{\alpha}{1-\alpha}} 100000$$

Question 3.

Consider an economy with the production function:

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{0.5} E^{1-0.5}$$

in which the depreciation rate on capital is three percent per year, the rate of population growth is one percent per year, and the rate of growth of labor-augmenting technology is one percent per year.

a. Suppose that the savings rate is ten percent of GDP. What is the steady-state capital-output ratio? What is the value of output per worker on the steady-state growth path written as a function of the level of labor-augmenting technology E?

$$\left(\frac{K}{Y}\right)_{SS} = \frac{s}{n+g+\delta} =$$

$$\frac{0.1}{0.01 + 0.01 + 0.03}$$

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2.
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$$\left(\frac{Y}{L}\right)_t = \left(\frac{K}{L}\right)_t^{0.5} E_t^{1-0.5} = \left(\left(\frac{K}{Y}\right)_t \left(\frac{Y}{L}\right)_t\right)^{0.5} E_t^{1-0.5} = \left(\frac{K}{Y}\right)_t^{0.5} \left(\frac{Y}{L}\right)_t^{0.5} E_t^{1-0.5}$$

$$\Rightarrow \left(\frac{Y}{L}\right)_t^{0.5} = \left(\frac{K}{Y}\right)_t^{0.5} E_t^{1-0.5}$$

$$\Rightarrow \left(\frac{Y}{L}\right)_t = \left(\frac{K}{Y}\right)_t^{\frac{0.5}{1-0.5}} E_t = \left(\frac{K}{Y}\right)_t E_t$$

$$\Rightarrow \left(\frac{Y}{L}\right)_{ss,t} = \left(\frac{K}{Y}\right)_{ss} E_t = 2\, E_t$$

b. Suppose that the savings rate is fifteen percent of GDP. What is the steady-state capital-output ratio? What is the value of output per worker on the steady-state growth path written as a function of the level of labor-augmenting technology E?

$$\left(\frac{K}{Y}\right)_{SS} = \frac{s}{n+g+\delta} =$$

$$\frac{0.15}{0.01 + 0.01 + 0.03}$$

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3.
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$$\left(\frac{Y}{L}\right)_{ss,t} = \left(\frac{K}{Y}\right)_{ss} E_t = 3\,E_t$$

c. Suppose that the savings rate is twenty percent of GDP. What is the steady-state capital-output ratio? What is the value of output per worker on the steady-state growth path written as a function of the level of labor-augmenting technology E?

$$\left(\frac{K}{Y}\right)_{SS} = \frac{s}{n+g+\delta} =$$

$$\frac{0.20}{0.01 + 0.01 + 0.03}$$

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4.
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$$\left(\frac{Y}{L}\right)_{ss,t} = \left(\frac{K}{Y}\right)_{ss} E_t = 4\,E_t$$

In conclusion, an increase in the savings rate always increases output per worker. Of course the maximum you can save is 100% of your income and an increase in ooutput per worker does not necessarily imply an increase in consumption per worker (cf. Golden Rule.)

Question 4.

What happens to the steady-state capital-output ratio if the rate of technological progress increases? Would the steady-state growth path of output per worker for the economy shift upward, downward, or remain in the same position?

$$\partial_g \left(\frac{K}{Y}\right)_{ss} =$$

$$\partial_g \left(\frac{s}{n+g+\delta}\right)$$

$$-\frac{s}{(g+n+\delta)^2}$$

The steady-state capital-output ratio goes down if the rate of technological progress increases and since $\left(\frac{Y}{L}\right)_{ss,t} = \left(\frac{K}{Y}\right)_{ss} E_t$, the steady – state growth path of output per worker for the economy shifts downward

To see why recall that :

$$\frac{\left(\frac{\dot{K}}{Y}\right)_t}{\left(\frac{K}{Y}\right)_t} = 0 \Leftrightarrow \frac{\dot{K}_t}{K_t} = n + g$$

Thus when the rate of technical progress is faster, the capital stock needs to grow faster to maintain the steady-state capital-output ratio. But that implies that:

$$\dot{K}_t = sY_t - \delta K_t = (n+g) K_t \Leftrightarrow sY_t - (n+g) K_t = \delta K_t$$

In steady state, $sY_t - (n+g) K_t$, gross investment per unit of efficient labor has to match δK_t, depreciation. With dimishing marginal returns to capital, it's going to be harder for gross investment to make up for depreciation when the number of units of efficient labor is growing faster. Therefore a lower of capital – output ratio will be sustainable in steady – state.

However, since $\frac{\dot{E}_t}{E_t} = g$,

the steady – state growth path of output per worker is also more markedly curved upwards, i.e. output per worker will grow faster.

Question 5.

Discuss--that is, write two paragraphs evaluating--the following proposition: "An increase in the savings rate will increase the steady-state capital output ratio, and so increase both output per worker and the rate of economic growth both in the first few years after the savings rate has increased and in the very long run as well."

An increase in the savings rate will definitely increase the steady-state output ratio since it will allow for higher gross investment. This will raise the steady-state growth path for output per worker.

During the transition to the new steady-state, output per worker will grow faster as the capital-output ratio converges to its new higher steady-state. However, in the long-run, as the savings rate as no effect on the rate of technological progress, output per capita will be growing at that latter rate.

A change in the savings rate thus has a level effect but no growth effect on output per capita in the long-run.

Question 6.

Would the steady-state growth path of output per worker for the economy shift upward, downward, or remain the same if capital were to become more durable--if the rate of depreciation on capital were to fall?

$$\partial_\delta \left(\frac{K}{Y}\right)_{ss} =$$

$$\partial_\delta \left(\frac{s}{n + g + \delta}\right)$$

$$-\frac{s}{(g + n + \delta)^2}$$

The steady-state capital-output ratio goes down if capital were to become more durable, *i.e.* if δ increases, and since $\left(\frac{Y}{L}\right)_{ss,t} = \left(\frac{K}{Y}\right)_{ss} E_t$, the steady – state growth path of output per worker for the economy shifts downward

Obviously, if capital depreciates faster, it's going to be harder for gross investment of unit of efficient labor to make up for depreciation. Thus, a lower capital-output ratio will be reached in the new steady-state.

Question 7

Suppose that a sudden disaster--an epidemic, say--reduces a country's population and labor force, but does not affect its capital stock. Suppose further that the economy was on its steady-state growth path before the epidemic. What is the immediate effect of the epidemic on output per worker? On the total economy-wide level of output? What happens subsequently?

Recall that $\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha} E^{1-\alpha}$; thus if the labor force decreases, the capital – labor ratio increases which results in an increase in output per worker. Indeed, all of a sudden, workers find themselves equipped with more machines, they are therefore more productive.

However, since $Y=K^{\alpha}(LE)^{1-\alpha}$, when the labor force decreases, total output decreases. Less of any input, always means less production, the marginal productivity of factors of production is always positive (even though it's decreasing.)

But now, $\frac{K}{Y} > \left(\frac{K}{Y}\right)_{SS}$ since total output has decreased, the capital stock has not chnaged yet and none of the determinants of the steady state capital output ratio are assumed to have changed. By the definition of a steady state, if the capital – output ratio is higher than its steady state value, it will converge back to its steady state value, i.e. $\frac{(\dot{K/Y})}{K/Y} < 0$.

But since $\frac{Y}{L} = \left(\frac{K}{y}\right)^{\frac{\alpha}{1-\alpha}} E$, output per worker will also start to converge back to its original steady state growth path.

- QUESTION: Make sure you can graph what's going on here (check your section notes afterwards.)

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