

Economics 202a, Fall 1998

Lecture 2: Solow Model Handout

Basics:

$$Y_t = F(K_t, A_t L_t)$$

$$\frac{dA_t}{dt} = gA_t$$

$$\frac{dL_t}{dt} = nL_t$$

The capital-accumulation equation is more interesting:

$$\frac{dK_t}{dt} = sY_t - \delta K_t$$

Now we are in the simplification business, so we need to simplify: we want a system with only one variable.

So we assume constant returns to scale, so that:

$$\frac{Y_t}{A_t L_t} = \frac{1}{A_t L_t} F(K_t, A_t L_t) = F\left(\frac{K_t}{A_t L_t}, 1\right)$$

and we adopt so-called “intensive” notation with lower-case letters for values of variables per unit of “effective labor”:

$$\frac{Y_t}{A_t L_t} = y_t = F\left(\frac{K_t}{A_t L_t}, 1\right) = F(k_t, 1) = f(k_t)$$

We can use the identities:

$$\frac{d}{dt} \ln(x) = \frac{1}{x} \frac{dx}{dt}, \ln(xy) = \ln(x) + \ln(y)$$

to calculate the accumulation equation for $k = K/(AL)$:

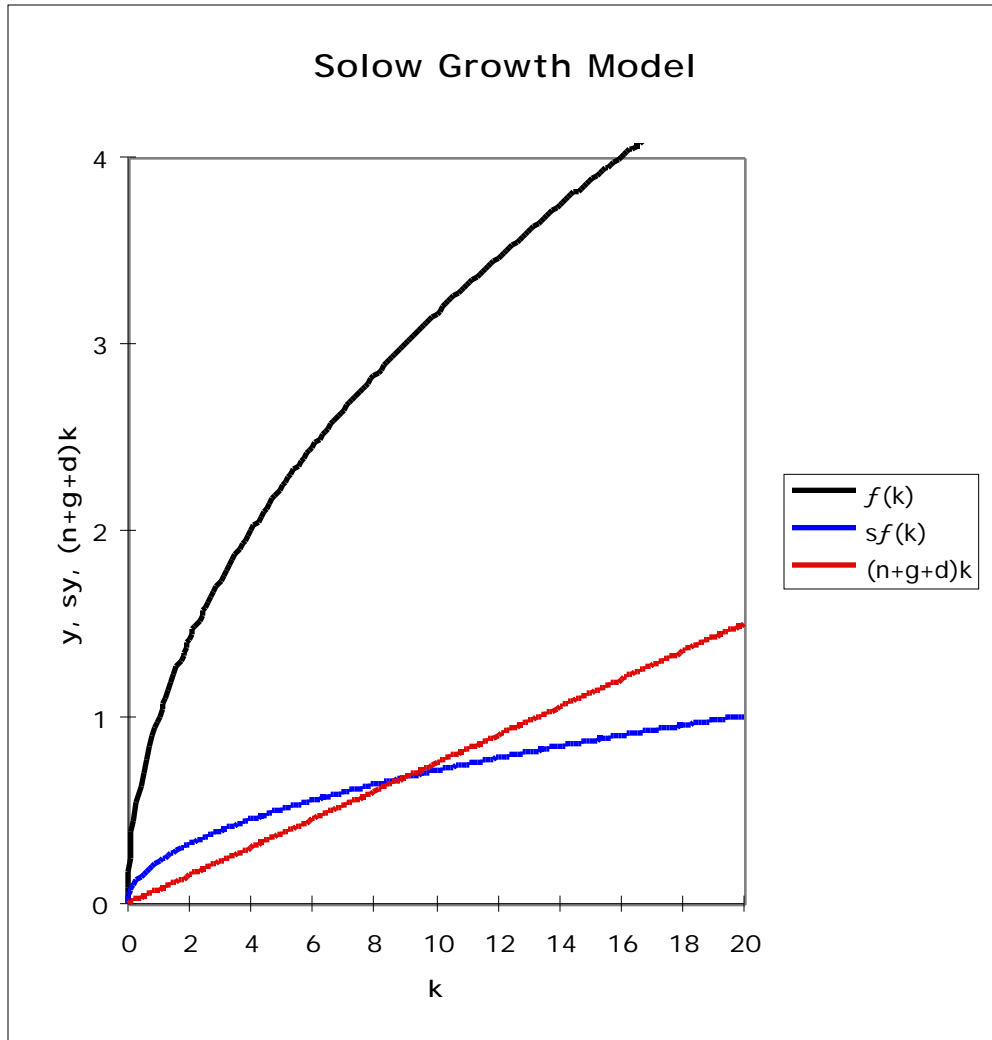
$$\frac{d}{dt} \ln(k_t) = \frac{d}{dt} \ln(K_t) - \frac{d}{dt} \ln(A_t) - \frac{d}{dt} \ln(L_t)$$

$$\frac{1}{k_t} \frac{dk_t}{dt} = s \frac{Y_t}{K_t} - \delta - g - n$$

$$\frac{1}{k_t} \frac{dk_t}{dt} = s \frac{y_t}{k_t} - \delta - g - n$$

$$\frac{dk_t}{dt} = sf(k_t) - (n + g + \delta)k_t$$

And now we can draw pretty diagrams...



And we can also draw some ugly diagrams. But we want to rule those out. So let's require:

$$f'(0) = \infty$$

$$f(0) = 0$$

$$f(x) > 0, \quad x \in (0, \infty)$$

$$f''(x) < 0, \quad x \in (0, \infty)$$

And so now we can only draw pretty diagrams.

And we can also note that there is a k^* such that:

$$(k_t < k^*) \quad \frac{dk_t}{dt} > 0$$

$$(k_t > k^*) \quad \frac{dk_t}{dt} < 0$$

$$(k_t = k^*) \quad \frac{dk_t}{dt} = 0$$

Steady-state growth path:

$$y^* = f(k^*)$$

$$Y_t^* = A_t L_t y^*$$

$$K_t^* = A_t L_t k^*$$

$$\frac{K_t^*}{Y_t^*} = z^* = \frac{k^*}{y^*}$$

$$\frac{Y_t^*}{L_t} = A_t y^*$$

$$\frac{K_t^*}{L_t} = A_t k^*$$

Cobb-Douglas Case:

We can actually get considerably further if we want to restrict ourselves to a smaller class of production functions. Say, the Cobb-Douglas case:

$$F(K_t, A_t L_t) = (K_t)^\alpha (A_t L_t)^{1-\alpha}$$

Then the “intensive” form is particularly simple:

$$\frac{Y_t}{A_t L_t} = \frac{1}{A_t L_t} (K_t^\alpha (A_t L_t)^{1-\alpha}) = \frac{K_t}{A_t L_t}^\alpha \frac{A_t L_t}{A_t L_t}^{1-\alpha}$$

$$y_t = (k_t)^\alpha$$

Then we know that:

$$\frac{d}{dt} \ln(k_t) = \frac{sf(k_t)}{k_t} - (n + g + \delta)$$

$$\frac{d}{dt} \ln(k_t) = s k_t^{\alpha-1} - (n + g + \delta)$$

$$\frac{d}{dt} \ln(y_t) = \frac{d}{dt} \ln(k_t^\alpha) = \frac{d}{dt} \alpha \ln(k_t) = \alpha \frac{d}{dt} \ln(k_t)$$

So let's write $z = k/y = K/Y$ for the capital-output ratio, and see that:

$$\frac{d}{dt} \ln(z_t) = \frac{d}{dt} \ln(k_t) - \frac{d}{dt} \ln(y_t) = (1 - \alpha) \frac{d}{dt} \ln(k_t)$$

$$\frac{d}{dt} \ln(z_t) = (1 - \alpha) (s k_t^{\alpha-1} - (n + g + \delta))$$

$$\frac{d}{dt} \ln(z_t) = (1 - \alpha) (s z_t^{-1} - (n + g + \delta))$$

$$\frac{dz_t}{dt} = (1 - \alpha) (s - (n + g + \delta) z_t)$$

And this is our goal: we can write:

$$z_t = \frac{s}{n + g + \delta} + \varepsilon_t$$

$$\frac{d\varepsilon_t}{dt} = (\alpha - 1)(n + g + \delta)\varepsilon_t$$

$$\varepsilon_t = A_0 d^{(\alpha - 1)(n + g + \delta)t}$$

$$z_t = \frac{s}{n + g + \delta} + A_0 e^{(\alpha - 1)(n + g + \delta)t}$$

$$\frac{K_t}{Y_t} = \frac{k_t}{y_t} = z_t = \frac{s}{n + g + \delta} + \frac{K_0}{Y_0} - \frac{s}{n + g + \delta} e^{(\alpha - 1)(n + g + \delta)t}$$

And with this we can calculate, especially once we realize that:

$$Y_t = K_t^\alpha (A_t L_t)^{1 - \alpha} = \frac{K_t}{Y_t}^\alpha Y_t (A_t L_t)^{1 - \alpha}$$

$$Y_t = Y_t^\alpha (z_t)^\alpha (A_t L_t)^{1 - \alpha}$$

$$Y_t = A_t L_t (z_t)^{\frac{\alpha}{1 - \alpha}}$$

$$\frac{Y_t}{L_t} = A_t (z_t)^{\frac{\alpha}{1 - \alpha}}$$

$$y_t = (z_t)^{\frac{\alpha}{1 - \alpha}}$$