Investment with Uncertainty and Asymmetric Adjustment Costs

Paul Krugman's version of Avinash Dixit's model…

> Paul Krugman (1989), *Exchange Rate Instability* (Cambridge: MIT Press: 0262111403).
> Avinash Dixit (1989), "Entry and Exit Decisions of a Firm Under Fluctuating Exchange Rates," *Journal of Political Economy*.

Begin with a firm that must spend an irrecoverable fixed cost F to enter a market (but that can drop out of the market at cost zero by abandoning its assets). While the firm is in the market, it earns operating profits:

$$\pi_t = e^{p_t} - 1$$

where Dixit interprets e^p as an exchange rate (and the firm as an exporter selling into some foreign market), but other interpretations are valid as well. And the firm discounts future profits at a (risk-neutral) interest rate r.

We assume that p follows a *diffusion process*, or a *continuous time random walk*. That is, we say that for any time interval Δt (no matter how short):

$$E_t(p_{t+\Delta t}) = p_t$$

$$E_t\left[(p_{t+\Delta t} - p_t)^2\right] = \sigma^2 \Delta t$$

Given this environment—with the demand for its product random-walking all over the place—the firm has to decide (a) when to invest F and enter the market, and (b) when to exit.

We solve this by defining two value functions, representing the value to a firm that behaves optimally of being in and out of the market as a function of the current value of the demand variable p. We start to figure out what these value functions are by nailing down some boundary conditions. The firm will abandon its capital and exit the market at some p^O at which:

$$V_O(p^O) = V_I(p^O)$$

The firm will invest the fixed cost F and enter the market at some p^I at which:

$$V_I(p^I) = V_O(p^I) + F$$

And we continue the process of determining the value function by the standard procedure of thinking of the value of being in the state I or O as an asset that must yield a competitive return:

$$rV_I(p_t) = (e^{p_t} - 1) + E_t\left\{\frac{dV_I(p_t)}{dt}\right\}$$

$$rV_O(p_t) = E_t\left\{\frac{dV_O(p_t)}{dt}\right\}$$

But what are these expectations? Well...

$$E_t\left\{\frac{V_I(p_{t+\Delta t}) - V_I(p_t)}{\Delta t}\right\} = E_t\left\{\frac{V'_I(p_t)(\Delta p) + (1/2)V''_I(p_t)(\Delta p)^2 + \ldots}{\Delta t}\right\} =$$

$$\frac{V'_I(p_t)E_t(\Delta p) + (1/2)V''_I(p_t)E_t(\Delta p)^2 + \ldots}{\Delta t} = \frac{(1/2)V''_I(p_t)\sigma^2(\Delta t) + \ldots}{\Delta t} =$$

$$(1/2)V''_I(p_t)\sigma^2$$

in the limit as. Since the derivative is defined as the limit of the quotient above as Δt approaches zero, I assert that:

$$E_t\left\{\frac{dV_I(p_t)}{dt}\right\} = (\sigma^2/2)V''_I(p_t)$$

$$E_t\left\{\frac{dV_O(p_t)}{dt}\right\} = (\sigma^2/2)V''_O(p_t)$$

And the differential equations for the value functions as functions of the state variable p are:

$$V_I(p_t) = \frac{e^{p_t} - 1}{r} + \frac{\sigma^2 V''_I(p_t)}{2r}$$

$$V_O(p_t) = \frac{\sigma^2 V''_O(p_t)}{2r}$$

Let me write down the general solutions:

$$V_I(p_t) = \frac{e^{p_t}}{r - \sigma^2/2} - \frac{1}{r} + Ae^{\sqrt{\frac{2r}{\sigma^2}}p_t} + Be^{-\sqrt{\frac{2r}{\sigma^2}}p_t}$$

$$V_O(p_t) = Ce^{\sqrt{\frac{2r}{\sigma^2}}p_t} + De^{-\sqrt{\frac{2r}{\sigma^2}}p_t}$$

And invoke boundary conditions—that when p is large, the value of being in should be "near" its fundamental value; and that when p is very very small, the value of being out should be "near" zero—to claim that A and D must be zero. So we have:

$$V_I(p_t) = \frac{e^{p_t}}{r - \sigma^2/2} - \frac{1}{r} + Be^{-\sqrt{\frac{2r}{\sigma^2}}p_t}$$

$$V_O(p_t) = Ce^{\sqrt{\frac{2r}{\sigma^2}}p_t}$$

And we have our boundary conditions for the states at which the firm enters and exits the market:

$$V_I(p^I) = V_O(p^I) + F$$

$$V_O(p^O) = V_I(p^O)$$

But we have four unknowns—B, C, p^O, and p^I—and only two equilibrium conditions. We need two more: "smooth pasting" conditions that come from the requirement that p^I and p^O be not just the points at which the firm switches from being out to in and from in to out, but that they be the *optimal* points at which to enter and exit. Not only must the value functions be equal at those two points, they must be tangent:

$$V'_I(p^I) = V'_O(p^I)$$
$$V'_I(p^O) = V'_O(p^O)$$