Suggested Solutions to Problem Set 3

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(a) **Capital Levy** The effect of the one-time capital levy is to increase the value of capital at time t_2. In particular, the value of capital an instant $\varepsilon > 0$ before the levy equals the value of capital after the levy minus the amount of the capital levy:

$$q(t_2 - \varepsilon) = q(t_2 + \varepsilon) - f \cdot q\,(t_2 - \varepsilon)\,,$$

or equivalently, $(1 + f) \cdot q(t_2 - \varepsilon) = q(t_2 + \varepsilon)$. Thus, at time t_2 q jumps up by $f \cdot 100\%$ from B to C and the economy lands on the saddle path towards the initial equilibrium denoted by E (see figure 1).

- **(i)** At the time of the announcement, t_1, the value of capital q will drop in anticipation of the capital levy. The economy jumps from E to A. The rate of investment I slumps as firms start decumulating capital in response to the drop in q.
- **(ii)** Between t_1 and t_2, the economy obeys the dynamics of the system and moves gradually to B. The value of capital declines as the capital levy becomes more eminent; the rate of investment drops correspondingly and the capital stock shrinks.
- **(iii)** At the time of the capital levy, t_2, the value of capital jumps by $100f$ percent, from B to C. Subsequently, the economy moves along the saddle path towards the initial equilibrium E. The value of capital is enhanced by the low level of the capital stock. This makes investment attractive and raises the capital stock back to its initial level.
- **(iv)** Clearly, the level of q jumps discontinuously twice. First at the time of the announcement, t_1, and then at the time of the capital levy, t_2.

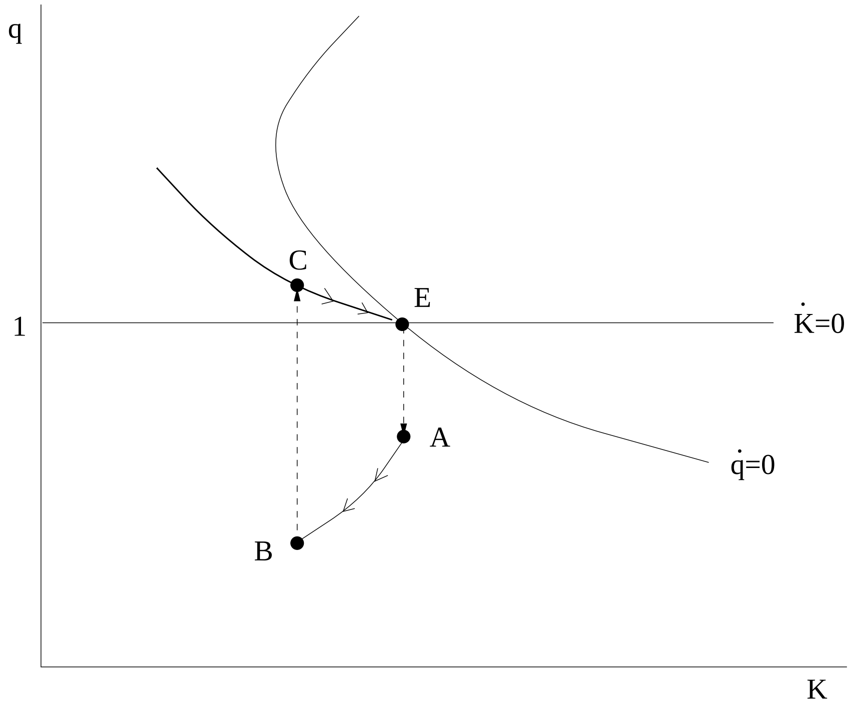

Figure 1: The effect of a capital levy

(b) Investment and Real Wages Denote the firm's profits in period s by π_s, so that

$$\pi_s = AK_s^a L_s^{1-a} - w_s L_s - I_s - bI_s^2. \tag{1}$$

(i) In each period s, the optimal level of employment is given by $\partial \pi_s / \partial L_s = (1-a)AK_s^a L_s^{-a} - w_s = 0$, which amounts to

$$L_s = \left[\frac{(1-a)A}{w_s}\right]^{\frac{1}{a}} K_s. \tag{2}$$

Substituting (2) into (1) and simplifying yields

$$\begin{aligned} \pi_s^* &= (1-a)^{\frac{1-a}{a}} A^{\frac{1}{a}} w_s^{-\frac{1-a}{a}} K_s - (1-a)^{\frac{1}{a}} A^{\frac{1}{a}} w_s^{-\frac{1-a}{a}} K_s - I_s - bI_s^2 \\ &= a(1-a)^{\frac{1-a}{a}} A^{\frac{1}{a}} w_s^{-\frac{1-a}{a}} K_s - I_s - bI_s^2. \end{aligned}$$

(ii) The firm maximizes $\Pi_t = \mathrm{E}_t \left[\sum_{s=t}^{\infty} e^{-r(s-t)} \pi_s^*\right]$ subject to the capital accumulation equation $K_{s+1} - K_s = I_s - dK_s$, for $s = t, t+1, t+2, ...$ Hence, the Lagrangian equals

$$
\begin{aligned}
\mathcal{L} &= \mathrm{E}_t \left\{ \sum_{s=t}^{\infty} e^{-r(s-t)} \left[a(1-a)^{\frac{1-a}{a}} A^{\frac{1}{a}} w_s^{-\frac{1-a}{a}} K_s - I_s - bI_s^2 \right] \right\} \\
&\quad + \sum_{s=t}^{\infty} \lambda_s \left[(1-d)K_s + I_s - K_{s+1} \right].
\end{aligned}
$$

The first-order conditions are

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial I_t} &= -1 - bI_t + \lambda_t = 0 && (3) \\
\frac{\partial \mathcal{L}}{\partial K_{t+1}} &= -\lambda_t + \mathrm{E}_t \left[e^{-r} a(1-a)^{\frac{1-a}{a}} A^{\frac{1}{a}} w_{t+1}^{-\frac{1-a}{a}} \right] + \lambda_{t+1}(1-d) = 0 && (4)
\end{aligned}
$$

Let us define $\xi_s \equiv e^{r(s-t)}\lambda_s$ as the 'current value' Lagrange multiplier, which is the undiscounted shadow price of capital at time s. Thus, ξ is the discrete-time analogue of Tobin's q. Using $\xi_t = \lambda_t$, (3) amounts to

$$
I_t = \frac{\xi_t - 1}{2b}. \qquad (5)
$$

In addition, define $f(w_s) \equiv a(1-a)^{\frac{1-a}{a}} A^{\frac{1}{a}} w_s^{-\frac{1-a}{a}}$. Then, taking the expected value as of time t, (4) can be written as

$$
\xi_t = e^{-r}\, \mathrm{E}_t\, f(w_{t+1}) + (1-d)e^{-r}\, \mathrm{E}_t\, \xi_{t+1}. \qquad (6)
$$

Similarly, $\xi_{t+k} = e^{-r}\, \mathrm{E}_{t+k}\, f(w_{t+k+1}) + (1-d)e^{-r}\, \mathrm{E}_{t+k}\, \xi_{t+k+1}$, for $k = 1, 2, ..., T$. Substituting recursively for ξ_{t+k} and using the law of iterated expectations, we obtain the forward solution

$$
\xi_t = \sum_{k=0}^{T} (1-d)^k e^{-r(k+1)}\, \mathrm{E}_t\, f(w_{t+k+1}) + (1-d)^T e^{-rT}\, \mathrm{E}_t\, \xi_{t+T}.
$$

Imposing the no-bubble condition, $\lim_{T\to\infty}(1-d)^T e^{-rT}\xi_{t+T} = 0$, we obtain an expression for the value of capital:

$$
\xi_t = \sum_{k=0}^{\infty} (1-d)^k e^{-r(k+1)}\, \mathrm{E}_t\, f(w_{t+k}). \qquad (7)
$$

Substituting (7) into (5), we have found optimal investment

$$
I_t = \frac{1}{2b} \sum_{k=1}^{\infty} (1-d)^{k-1} e^{-rk}\, \mathrm{E}_t\, f(w_{t+k}) - 2b.
$$

(iii) For an infinite horizon and a two-state Markov process for the wage w, the economy is either in state L with wage w_L and capital value q_L, or in state H with w_H and q_H. We can use (6) to write

$$\begin{aligned}
\xi_L &= pe^{-r}f(w_L)+(1-p)e^{-r}f(w_H) \\
&\quad +p(1-d)e^{-r}\xi_L+(1-p)(1-d)e^{-r}\xi_H, \\
\xi_H &= qe^{-r}f(w_H)+(1-q)e^{-r}f(w_L) \\
&\quad +q(1-d)e^{-r}\xi_H+(1-q)(1-d)e^{-r}\xi_L.
\end{aligned}$$

Conveniently solving this messy expression with Scientific WorkPlace gives

$$\begin{aligned}
\xi_L &= \frac{\left[(1-q-p)(1-d)e^{-2r}+pe^{-r}\right]f(w_L)+(1-p)e^{-r}f(w_H)}{1-(p+q)(1-d)e^{-r}-(1-p-q)(1-d)^2e^{-2r}} \\
\xi_H &= \frac{\left[(1-q-p)(1-d)e^{-2r}+qe^{-r}\right]f(w_H)+(1-q)e^{-r}f(w_L)}{1-(p+q)e^{-r}(1-d)-(1-p-q)e^{-2r}(1-d)^2}
\end{aligned}$$

Investment also follows a Markov process. In particular, using (5), $I_L=(\xi_L-1)/2b$ and $I_H=(\xi_H-1)/2b$. If the wage is currently w_L, then expected investment equals $pI_L+(1-p)I_H$; for w_H, expected investment is $qI_H+(1-q)I_L$. Changes in p and q have two effects on investment. They directly affect the state of the economy and thereby expected investment; and they indirectly affect the value of capital ξ_L and ξ_H and thereby I_L and I_H.

(c) Lucas Tree Asset Pricing The consumer maximizes the expected value of life-time utility $U=\sum_{s=t}^{\infty}(1+\rho)^{-(s-t)}\ln(C_s)$.

(i) Reducing current consumption by dC reduces utility by $\frac{1}{C_t}dC$. On the other hand, it allows the consumer to get $\frac{1}{P_t}dC$ extra trees, each of which produce $\mathrm{E}_t\, Y_{t+1}$ fruits and yield $\mathrm{E}_t\, P_{t+1}$ when sold in the next period; this will boost utility in the next period by $\mathrm{E}_t\,\frac{Y_{t+1}+P_{t+1}}{P_t}\frac{1}{C_{t+1}}dC$. The deviation dC has no first-order effect on expected life-time utility if

$$\frac{1}{C_t}=\frac{1}{1+\rho}\,\mathrm{E}_t\,\frac{Y_{t+1}+P_{t+1}}{P_t}\frac{1}{C_{t+1}} \tag{8}$$

(ii) Solving (8) for P_t and using the fact that $C_s=Y_s$ for $s=t,t+1,...$, gives

$$\begin{aligned}
P_t &= \frac{Y_t}{1+\rho}\,\mathrm{E}_t\,\frac{Y_{t+1}+P_{t+1}}{Y_{t+1}} \\
&= \frac{Y_t}{1+\rho}+\frac{Y_t}{1+\rho}\,\mathrm{E}_t\,\frac{P_{t+1}}{Y_{t+1}} \qquad (9)
\end{aligned}$$

(iii) Substituting $P_{t+1} = \frac{Y_{t+1}}{1+\rho} + \frac{Y_{t+1}}{1+\rho} \mathrm{E}_{t+1} \frac{P_{t+2}}{Y_{t+2}}$ into (9) and using iterated expectations,

$$\begin{aligned} P_t &= \frac{Y_t}{1+\rho} + \frac{Y_t}{1+\rho} \mathrm{E}_t \frac{P_{t+1}}{Y_{t+1}} \\ &= \frac{Y_t}{1+\rho} + \frac{Y_t}{(1+\rho)^2} + \frac{Y_t}{(1+\rho)^2} \mathrm{E}_t \frac{P_{t+2}}{Y_{t+2}}. \end{aligned}$$

Subsequent forward iteration produces

$$P_t = \sum_{s=t+1}^{T} \frac{Y_t}{(1+\rho)^{s-t}} + \frac{Y_t}{(1+\rho)^{T-t}} \mathrm{E}_t \frac{P_{t+T}}{Y_{t+T}}.$$

Ruling out speculative bubbles, the second term on the right-hand side vanishes as $T \to \infty$. Hence,

$$P_t = Y_t \sum_{s=t+1}^{\infty} \frac{1}{(1+\rho)^{s-t}} = Y_t \frac{\frac{1}{1+\rho}}{1 - \frac{1}{1+\rho}} = \frac{1}{\rho} Y_t. \tag{10}$$

(iv) It follows from (10) that the price of Lucas' tree is not affected by the expected value of future dividends. The reason is that there are two opposing forces. The first is the fact that higher expected future dividends make trees more attractive, thereby raising the current price of a tree. But, higher future dividends amount to higher future consumption and thus lower marginal utility. The fact that the tree will produce more fruits when marginal utility is low reduces the attractiveness of owning the tree, thereby depressing the current price. In the case of logarithmic utility, these two effects exactly offset each other, leaving the current price of the tree unchanged when expected future dividends increase.

(d) The q-theory model Recall that the dynamics in the phase diagram are described by

$$\begin{aligned} \dot{q} &= rq - F_K(K, \bar{L}) - \frac{(q-1)^2}{2\chi} \\ \dot{K} &= \frac{q-1}{\chi} K. \end{aligned}$$

(i) A tax t on income from capital shifts the $\dot{q} = 0$ locus to the left as $q = \left[(1-t)F_K + (q-1)^2/2\chi\right]/r$. The economy jumps from the initial equilibrium E to A on the saddle path towards the new steady state E'

(see figure 2). The value of capital q falls discontinuously at the time of the capital-income tax. This leads to disinvestment, which reduces the capital stock K. The lower levels of capital increase the market value of capital until q returns to its equilibrium value of one. The capital stock is permanently reduced.

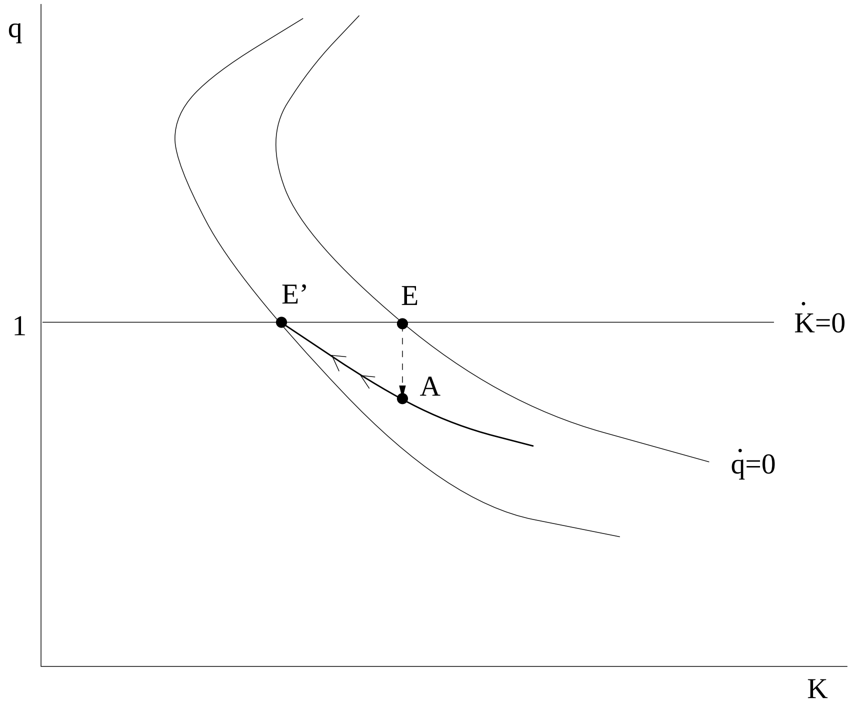

Figure 2: The effect of a capital-income tax

(ii) If a war destroys half of the capital stock from K^* to K_0, then the remaining capital becomes more valuable. So, q jumps up, putting the economy at A, on the saddle path towards the initial equilibrium E (see figure 3). The higher market value of capital encourages investment. The capital stock expands until it reaches its initial level K^* at a value of $q = 1$.

(iii) An investment tax credit of g affects the cost of acquiring capital. In particular, the value of capital now equals $q = 1 - g + C_I$. Hence, the new $\dot{K} = 0$ locus shifts down to the new equilibrium value of capital $1 - g$. When the investment tax credit is suddenly enacted, the value of capital drops discontinuously from E to A (see figure 4). The new level of q exceeds $1 - g$, so it boosts investment and starts increasing the capital

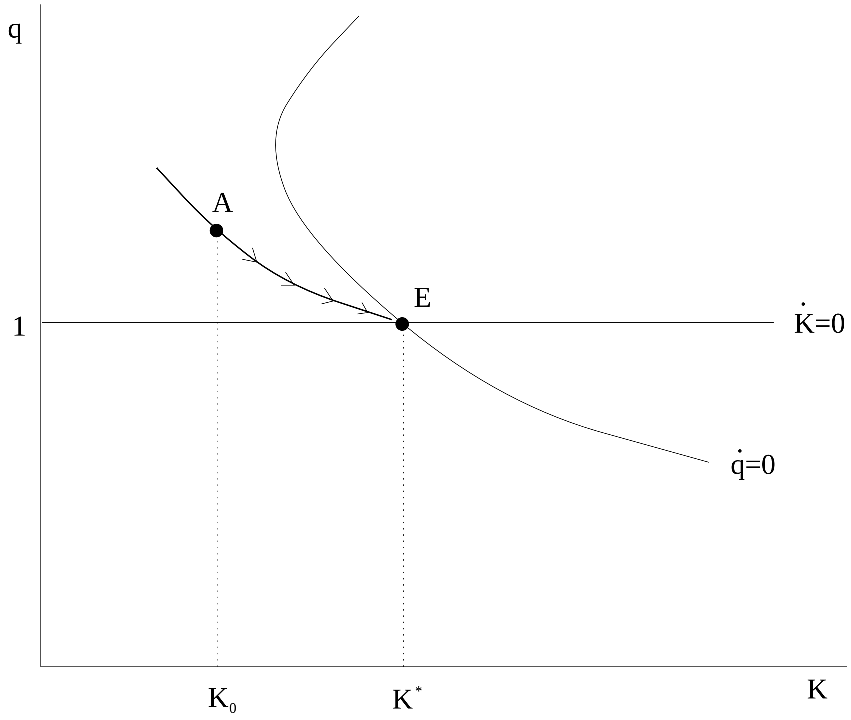

Figure 3: A destruction of the capital stock

stock. As a result, the economy moves along the saddle path towards the new equilibrium E' with a permanently higher capital stock.

(iv) The announcement of the investment tax credit at time t_1 induces a drop in the value of capital in anticipation of the tax credit. The economy jumps from E to A (see figure 5). Still obeying the dynamics around the initial equilibrium E, the value of capital decreases further, which discourages investment and reduces the capital stock. Intuitively, firms postpone investment to capture the benefits of the investment tax credit at time t_2. When the tax credit is enacted, the economy has arrived at B. Now q is above its new equilibrium value of $1 - g$. So, firms start investing more and the capital stock expands as the economy moves along the saddle path towards the new equilibrium E' at a permanently higher capital stock.

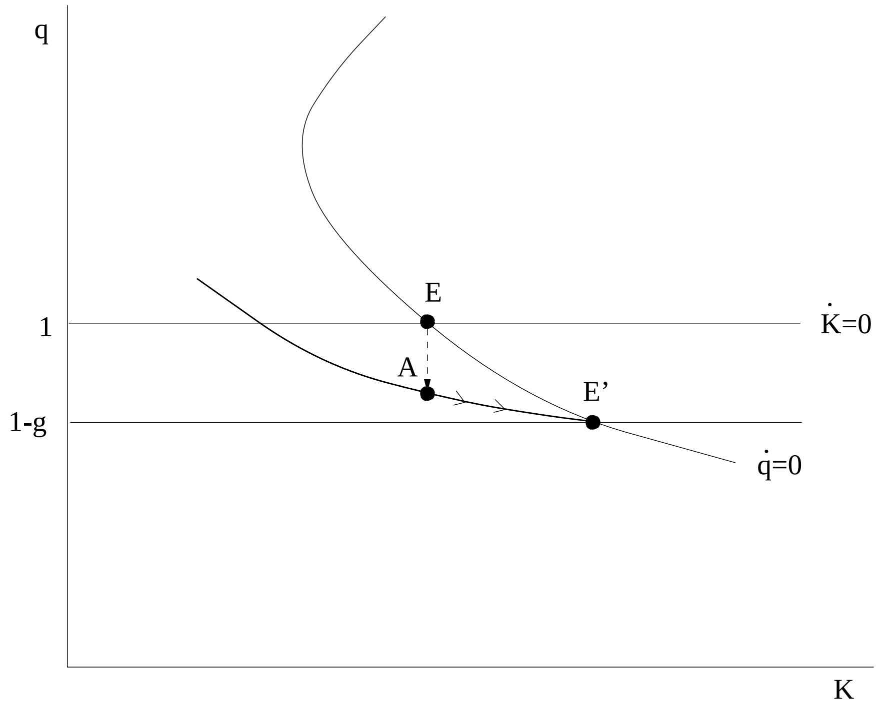

Figure 4: A sudden investment tax credit

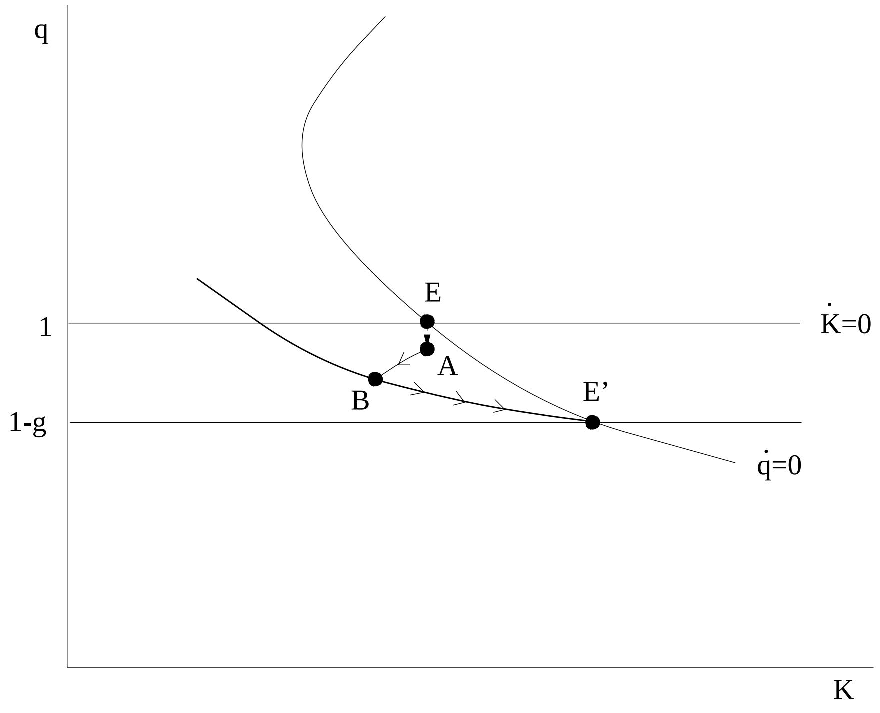

Figure 5: An anticipated investment tax credit